Solution to the Midterm Exam

Math 221

Feb. 23, 2007

1. Consider the differential equation
$$x^2u''(x) + xu'(x) - u(x) = 0.$$
   (a) (2 points) Is $u(x) = \sin x$ a solution?
   We differentiate: $u' = \cos x$ and $u'' = \sin x$. Thus
$$x^2u'' + xu' - u = -x^2 \sin x + x \cos x - \sin x \neq 0,$$
   and so $u = \sin x$ is not a solution.
   (b) (2 points) Is $u(x) = x + 1/x$ a solution?
   Again we differentiate: $u' = 1 - x^{-2}$ and $u'' = 2x^{-3}$, and so
$$x^2u'' + xu' - u = \frac{2}{x} + x - \frac{1}{x} - x - \frac{1}{x} = 0,$$
   and so $u = x + 1/x$ is a solution.
2. (10 points) Solve the initial value problem
$$u' + 2xu = x, \qquad u(0) = -1.$$
   This is a linear first order equation, so we multiply by the integrating factor
$$\rho = e^{\int 2x dx} = e^{x^2}.$$
   This transforms the equation into
$$\frac{d}{dx}(u(x)e^{x^2}) = xe^{x^2}.$$
   Now integrate both sides of the equation:
$$u(x)e^{x^2} = \int xe^{x^2}\, dx = \frac{1}{2}e^{x^2} + c \Rightarrow u(x) = \frac{1}{2} + ce^{-x^2}$$
   for some constant c. Now plug in the initial condition:
$$-1 = u(0) = \frac{1}{2} + c \Rightarrow c = -\frac{3}{2},$$
   and so
$$u = \frac{1}{2} - \frac{3}{2}e^{-x^2}.$$
   (Note: You can also separate variables in this equation, rewriting it as $u' = x(1 - 2u)$. Either method is fine.)
3. Consider the differential equation
$$u' = u \cos(u).$$
   (Note: you don't need to solve the differential equation.)
   (a) (5 points) Verify that the equilibria (*i.e.* the constant solutions) are $x = 0$ and half-integer multiples of π.
   The equilibria will satisfy
$$u' = 0 \Leftrightarrow u \cos(u) = 0.$$
   There are two possibilities: either $u = 0$ or $\cos(u) = 0$. In the second case,
$$\cos(u) = 0 \Leftrightarrow u == \frac{2n\pi + 1}{2}$$
   for some integer n, in other words u is a half integer multiple of π.

(b) (5 points) Characterize which of these equilibria are stable/unstable. Be sure to justify your answer.

We examine the phase line. Notice that the right hand side $u\cos(u)$ is positive for $0 < u < \pi/2$, negative for $\pi/2 < u < 3\pi/2$, and so on. In short, $u\cos(u)$ alternates sign as you cross each of its zeroes, starting positive immediately to the right of $u = 0$. Thus the equilibria alternate between stable and unstable, with $u = 0$ unstable and so on. For the other equilibria, we have that $u = (4n+1)\pi/2$ is stable for any integer n and $u = (4n+3)\pi/2$ is unstable for any integer n.

4. Consider the one–parameter family of differential equations

$$u' = u - hu^3.$$

(Note: you don't need to solve the differential equation.)

(a) (5 points) Find the equilibria. Your answer should depend on the parameter h.

Again, the equilibria will satisfy $u' = 0$, which gives us

$$0 = u - hu^3 = u(1 - hu^2).$$

If $h \leq 0$ the only (real) solution to this equation is $u = 0$. If $h > 0$, then we can factor the right hand side further as

$$u(1 - u\sqrt{h})(1 + u\sqrt{h}),$$

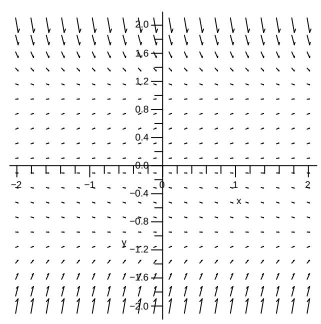
and so we get the solutions $u = 0, u = \pm 1/\sqrt{h}$. These are the equilibria.

(b) (5 points) This family of differential equations has a bifurcation point; find it. Be sure to explain your answer.

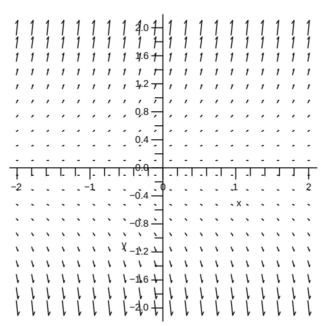
Notice that for $h \leq 0$ there is only one equilibrium ($u = 0$), but for $h > 0$ there are three equilibria: $u = 0, \pm 1/\sqrt{h}$. Thus $h = 0$ is the bifurcation point.

(c) (6 points) Sketch some typical solution curves for $h = 1$ and $h = -1$.

We look at $h = 1$ first. Observe that in this case the right hand side $u - u^3$ is positive for $u < -1$, negative for $-1 < u < 0$, positive for $0 < u < 1$, and negative for $u > 1$. Putting this all together, we see that the slope field looks like



Next we look at $h = -1$. In this case, the right hand side $u + u^3$ is negative for $u < 0$ and positive for $u > 0$. Thus the slope field looks like



5. Consider the differential equation

$$u'' - u' - 12u = 0.$$

(a) (6 points) Find the general solution to this equation. (Be sure to say something about why the general solution you write down can solve any initial value problem.)

We try solutions of the form $u = e^{rx}$:

$$0 = u'' - u' - 12u = e^{rx}(r^2 - r - 12) = e^{rx}(r-4)(r+3) \Rightarrow r = 4, -3.$$

So we have two solutions $u_1 = e^{4x}$ and $u_2 = e^{-3x}$. Also notice that these two functions are not multiples of each other, *i.e.* they are linearly independent. By the superposition principle we can write the general solution as

$$u = c_1 u_1 + c_2 u_2 = c_1 e^{4x} + c_2 e^{-3x}.$$

Another way to check that this is the general solution is to check that the Wronskian is nonzero:

$$W(0) = u_1(0)u_2'(0) - u_2(0)u_1'(0) = -3 - 4 = -7 \neq 0.$$

(b) (4 points) Solve the initial value problem for this differential equation with the initial conditions $u(0) = -1, u'(0) = 2$.

We have that $u = c_1 e^{4x} + c_2 e^{-3x}$, and want to match the initial condition $u(0) = -1, u'(0) = 2$. This leaves us with the system of linear equations

$$-1 = c_1 + c_2, \qquad 2 = 4c_1 - 3c_2.$$

Adding three times the first equation to the second we get

$$-1 = 7c_1 \Rightarrow c_1 = -\frac{1}{7}.$$

We plug this into either of the original equations to get $c_2 = -6/7$, and so the solution is

$$u(x) = -\frac{1}{7}e^{4x} - \frac{6}{7}e^{-3x}.$$