Practice Problems
Math 211
April 21, 2006

These problems are in no particular order.

1. Consider the differential equation
$$\frac{dx}{dt} = \frac{x}{t} + 2tx.$$
   (a) Is $x = \sin t$ a solution to this differential equation?
   (b) Is $x = te^{t^2}$ a solution to this differential equation?

2. Consider the differential equation
$$\frac{dx}{dt} = (e^x - 1)(1 - x^2).$$
   (a) Find all the equilibrium (i.e. constant) solutions.
   (b) Classify these equilibria as sinks, sources, or nodes.
   (c) Sketch the line field and some representative solution curves for this differential equation.

3. Consider the differential equation
$$\frac{dx}{dt} = (1 + 2t)(1 - x^2).$$
   (a) Find all the equilibrium solutions.
   (b) Find the general solution to the differential equation.
   (c) Solve the initial value problem with $x(0) = 1$.

4. Consider the one-parameter family of differential equations
$$\frac{dx}{dt} = ax - x^3.$$
   (a) Sketch the phase portrait when $a = 1$ and $a = -1$.
   (b) Verify that when $a = 1$ the only equilibria are $x = 0, 1, -1$. Also classify these equilibria as sinks, sources, or nodes.
   (c) Verify that when $a = -1$ the only equilibrium is $x = 0$, and classify this equilibrium as a snk, source, or node.
   (d) There is one bifurcation value for a. Find it.

5. Consider the system of differential equations
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
   (a) Find the eigenvalues of the coefficient matrix A.
   (b) Is the origin stable or unstable for this system? Be sure to explain your answer.
   (c) Find the associated eigenvectors and write down the general solution to the system.
   (d) Sketch the phase portrait of this system and some representative solution curves.
   (e) Solve the initial value problem with $x_1(0) = 1$ and $x_2(0) = -1$.

6. Consider the system of differential equations
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$
   (a) Find the eigenvalues of the coefficient matrix A.
   (b) Is the origin stable or unstable for this system? Be sure to explain your answer.
   (c) Find the associated eigenvectors and write down the general solution to the system.

(d) Sketch the phase portrait of this system and some representative solution curves.

(e) Solve the initial value problem with $x_1(0) = 0$ and $x_2(0) = 2$.

7. Consider the differential equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0.$$

(a) Find the general solution.

(b) Solve this initial value problem with $x(0) = -1$ and $x'(0) = 2$.

8. Consider the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 3e^{-2t}.$$

(a) Find the general solution to the associated homogeneous equation.

(b) Find a particular solution to the inhomogeneous equation, and combine this with your homogeneous solution to write out the general solution the inhomogeneous equation.

(c) Solve the initial value problem (for the inhomogeneous equation) with the initial conditions $x(0) = 2, x'(0) = 0$.

9. Consider the one-parameter family of differential equations with the parameter a:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} (e^x - 1)(a - y) \\ (e^y - 1)(a - x) \end{bmatrix}.$$

(a) Verfiy that the only equilibria of this system are $(0, 0)$ and (a, a).

(b) Linearize the system about both equilibria.

(c) Classify both these equilibria as stable or unstable.

(d) Sketch the phase potrait both when $a = 1$ and $a = -1$.

(e) There is one bifurcation point in a. Find it.

10. Consider the damped spring equation

$$x'' + \frac{1}{100}x' + 2x = 0$$

(a) Is this underdamped, overdamped, or critically damped?

(b) Sketch some representative solution curves (say with $x(0) > 0$ and $x'(0) = 0$).

11. Consider the forced spring equation

$$x'' + 9x = \cos(10t/3).$$

(a) What is the natural frequency of the spring? What is the forcing frequency?

(b) Find the frequency of the beats and the rapid oscillations.

(c) Sketch a representative solution curve. Be sure to label some points to indicate the scaling (e.g. the period of the beats).

12. Consider the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} x + x^2 - xy^2 \\ y - 2xy + y^2 \end{bmatrix}.$$

(a) Sketch the x and y nullclines for this system.

(b) Find the equilibrium solutions.

(c) Linearize the sytem about each equilibrium.

(d) Classify these equilibria as stable or unstable.