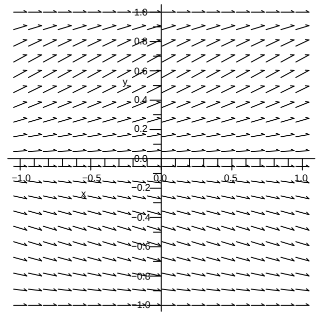
Solutions to the Practice Problems
Math 211
April 26, 2006

These problems are in no particular order.

1. Consider the differential equation
$$\frac{dx}{dt} = \frac{x}{t} + 2tx.$$
   (a) Is $x = \sin t$ a solution to this differential equation?
   We plug this function into both sides of the differential equation. First the left hand side:
$$\frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t,$$
   and next the right hand sides:
$$\frac{x}{t} + 2tx = \frac{\sin t}{t} + 2t \sin t.$$
   These two are not equal, so $x = \sin t$ is not a solution.
   (b) Is $x = te^{t^2}$ a solution to this differential equation?
   Again, we plug x into the differential equation, first on the left hand side:
$$\frac{dx}{dt} = \frac{d}{dx}(te^{t^2}) = e^{t^2} + 2t^2 e^{t^2},$$
   and next on the right hand side:
$$\frac{x}{t} + 2tx = \frac{te^{t^2}}{t} + 2t^2 e^{t^2} = e^{t^2} + 2t^2 e^{t^2}.$$
   The two sides are equal, so $x = te^{t^2}$ is a solution.
2. Consider the differential equation
$$\frac{dx}{dt} = (e^x - 1)(1 - x^2).$$
   (a) Find all the equilibrium (i.e. constant) solutions.
   The equilibria will satisfy
$$0 = \frac{dx}{dt} = (e^x - 1)(1 - x^2).$$
   This happens precisely when $e^x - 1 = 0$ or $1 - x^2 = 0$. In the first case, we must have $x = 0$, and in the second case we have $x = 1$ and $x = -1$. Thus the equilibria are $x = 0, 1, -1$.
   (b) Classify these equilibria as sinks, sources, or nodes.
   We test the value of $f(x) = (e^x - 1)(1 - x^2)$ at some points inbetween the equilibria:
$$f(1/2) = \frac{3}{4}(e^{1/2} - 1) > 0, \quad f(-1/2) = \frac{3}{4}(e^{-1/2} - 1) < 0, \quad f(3/2) = -\frac{5}{4}(e^{3/2} - 1) < 0, \quad f(-3/2) = -\frac{5}{4}(e^{-3/2} - 1) > 0.$$
   So we see that x is increasing for $x < -1$ and $0 < x < 1$, while x is decreasing for $-1 < x < 0$ and $x > 1$. Putting all this together, we see that $x = 0$ is a source while $x = 1$ and $x = -1$ are sinks.
   (c) Sketch the line field and some representative solution curves for this differential equation.
   Here is the line field:



3. Consider the differential equation

$$\frac{dx}{dt} = (1+2t)(1-x^2).$$

(a) Find all the equilibrium solutions.
The equilibria satisfy

$$0 = \frac{dx}{dt} = (1+2t)(1-x^2).$$

This holds for all values of t, and $t = 0$ in particular, thus we must have $1 - x^2 = 0$. In other words, $x = 1, -1$ are the equilibria.

(b) Find the general solution to the differential equation.
This is a separable equation:

$$\frac{dx}{dt} = (1+2t)(1-x^2) \Leftrightarrow \frac{dx}{1-x^2} = (1+2t)dt.$$

Integrating both sides of the equation, we have

$$t^2 + t + c = \int (1+2t)dt = \int \frac{dx}{1-x^2} = \int [\frac{1/2}{1-x} + \frac{1/2}{1+x}]dx = \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x).$$

Now solve for x:

$$t^2 + t + c = \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x) = \frac{1}{2}\log(\frac{1+x}{1-x}) = \log\sqrt{\frac{1+x}{1-x}},$$

which we can rearrange to read

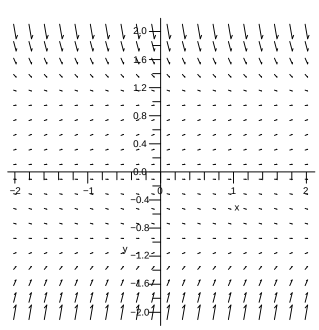
$$\frac{1+x}{1-x} = e^{2t^2+2t+c} \Leftrightarrow x = \frac{e^{2t^2+2t+c} - 1}{e^{2t^2+2t+c} + 1}.$$

(c) Solve the initial value problem with $x(0) = 1$.
The initial value is one of the equilibrium solutions, so the solution to the initial value problem is $x(t) = 1$ for all t.

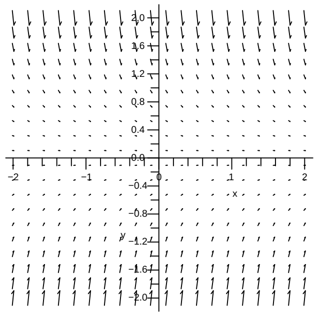
4. Consider the one-parameter family of differential equations

$$\frac{dx}{dt} = ax - x^3.$$

(a) Sketch the phase portrait when $a = 1$ and $a = -1$.
$a = 1$:



$a = -1$:



(b) Verify that when $a = 1$ the only equilibria are $x = 0, 1, -1$. Also classify these equilibria as sinks, sources, or nodes.

When $a = 1$, the differential equation reads $x' = x - x^3$, so the equilibria satisfy

$$0 = \frac{dx}{dt} = x - x^3 = x(1-x)(1+x) \Leftrightarrow x = 0, 1, -1.$$

To see if the equilibria are sinks, sources or nodes, we test some points: when $x = 1/2$ the righthand side is $1/2 - 1/8 > 0$, when $x = -1/2$ the right hand side is $-1/2 + 1/8 < 0$, when $x = 3/2$ the righthand side is $3/2 - 27/8 < 0$, and when $x = -3/2$ the righthand side is $-3/2 + 27/8 > 0$. Thus we see that x is increasing for $x < -1$ and $0 < x < 1$, while x is decreasing for $x > 1$ and $-1 < x < 0$, and so $x = 0$ is a source while $x = 1, -1$ are sinks.

(c) Verify that when $a = -1$ the only equilibrium is $x = 0$, and classify this equilibrium as a sink, source, or node.

When $a = -1$ the differential equation reads $x' = -x - x^2$, so so the equilibria satisfy

$$0 = \frac{dx}{dt} = -x - x^3 = -x(1 + x^2) \Leftrightarrow x = 0.$$

Again, we test some points to find whether $x = 0$ is a sink, source or node. When $x = 1$ the righthand side is $-2 < 0$, and when $x = -1$ the righthand side is $2 > 0$. Thus x is increasing for $x < 0$ and decreasing for $x < 0$, which means $x = 0$ is a sink.

(d) There is one bifurcation value for a. Find it.

We test for the number of equilibria:

$$0 = \frac{dx}{dt} = ax - x^3 = x(a - x^2).$$

When $a > 0$ this equation has three solutions: $x = 0, \pm\sqrt{a}$. On the other hand, when $a < 0$ the only solution is $x = 0$. Thus $a = 0$ is the bifurcation point.

5. Consider the system of differential equations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Find the eigenvalues of the coefficient matrix A.

The eigenvalues λ satisfy the equation

$$0 = \det \begin{bmatrix} \lambda + 1 & -3 \\ -5 & \lambda - 1 \end{bmatrix} = \lambda^2 - 9 = (\lambda + 3)(\lambda - 3),$$

so the eigenvalues are $\lambda = 3, -3$.

(b) Is the origin stable or unstable for this system? Be sure to explain your answer.

The origin is unstable because one of the eigenvalues is positive. Near the origin, the phase portrait looks like a saddle.

(c) Find the associated eigenvectors and write down the general solution to the system.

We first find the eigenvector for $\lambda = 3$. It satisfies the equation

$$\begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix} \Rightarrow b = \frac{4a}{3},$$

so we can choose the eigenvector to be

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Next we find the eigenvector for $\lambda = -3$:

$$\begin{bmatrix} -1 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3a \\ -3b \end{bmatrix} \Rightarrow b = -\frac{2a}{3},$$

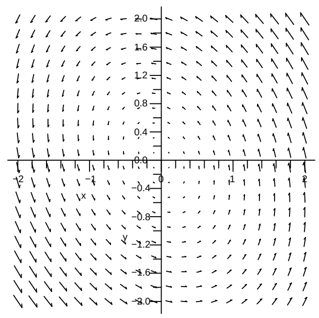
so we can choose the eigenvector to be

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Finally, we write the general solution as

$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

(d) Sketch the phase portrait of this system and some representative solution curves.



(e) Solve the initial value problem with $x_1(0) = 1$ and $x_2(0) = -1$.
We have to find the coeficients c_1 and c_2 by matching the initial conditions:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

We multiply the first equation by 2, multiply the second equation by 3, and add, to get $18c_1 = -1$, or $c_1 = -1/18$. Then plug this into either of the original equations to get $c_2 = 7/18$, so the solution to the initial value problem is

$$\vec{x}(t) = -\frac{1}{18} e^{3t} \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \frac{7}{18} \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

6. Consider the system of differential equations

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(a) Find the eigenvalues of the coefficient matrix A.
The eigenvalues λ satisfy the equation

$$0 = \det \begin{bmatrix} \lambda + 1 & 1 \\ -1 & \lambda + 1 \end{bmatrix} = \lambda^2 + 2\lambda + 2 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i.$$

(b) Is the origin stable or unstable for this system? Be sure to explain your answer.
The origin is stable (in fact, strictly stable) because the real parts of all the eigenvalues are negative. Near the origin, the solution curves look like stable spirals.

(c) Find the associated eigenvectors and write down the general solution to the system.
We first find the eigenvector for $\lambda = -1 + i$. It satisfies the equation

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (-1+i)a \\ (-1+i)b \end{bmatrix} \Rightarrow b = ia,$$

so we can choose the eigenvector to be

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

The eigenvector for $\lambda = -1 - i$ satisfies the equation

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (-1-i)a \\ (-1-i)b \end{bmatrix} \Rightarrow b = -ia,$$

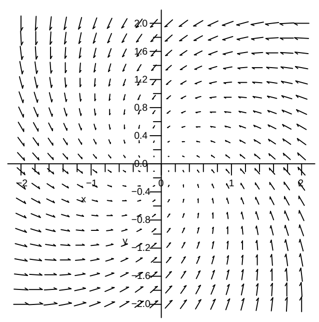
so we can choose the eigenvector to be

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Finally, this means we can write the general solution as

$$\vec{x}(t) = c_1 e^{(-1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(-1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

(d) Sketch the phase portrait of this system and some representative solution curves.



(e) Solve the initial value problem with $x_1(0) = 0$ and $x_2(0) = 2$.
We have to find coefficients c_1 and c_2 to match the initial conditions. They satisfy the equation

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

This system reads $c_1 + c_2 = 0, ic_1 - ic_2 = 2$. Sustituting $c_2 = -c_1$ into the second equation, we get $2ic_1 = 2$, or $c_1 = -i$. This in turn implies $c_2 = i$, so the solution to the initial value problem is

$$x(t) = -ie^{(-1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + ie^{(-1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

7. Consider the differential equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0.$$

(a) Find the general solution.
We look for solutions of the form $x(t) = e^{rt}$, then

$$0 = x'' - 3x' + 2x = r^2 e^{rt} - 3re^{rt} + 2e^{rt} = e^{rt}(r^2 - 3r + 2) = e^{rt}(r-1)(r-2),$$

so we must have $r = 1$ or $r = 2$. Thus the general solution is

$$x(t) = c_1 e^t + c_2 e^{2t}.$$

(b) Solve this initial value problem with $x(0) = -1$ and $x'(0) = 2$.
We have to find the coefficients c_1 and c_2 by matching initial conditions:

$$-1 = x(0) = c_1 + c_2, \qquad 2 = x'(0) = c_1 + 2c_2.$$

Subtracting the two equations, we get $c_2 = 3$, which we can then plug back in to either equation to get $c_1 = -4$. Thus the solution to the initial value problem is

$$x(t) = -4e^t + 3e^{2t}.$$

8. Consider the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 3e^{-2t}.$$

(a) Find the general solution to the associated homogeneous equation.
The homogeneous equation is $x'' + 5x' + 6x = 0$, and we look for solutions of the form $x = e^{rt}$. Then we must have

$$0 = r^2e^{rt} + 5re^{rt} + 6e^{rt} = e^{rt}(x + 2)(x + 3),$$

so $r = -2, -3$ and the general solution to the homogeneous equation is $c_1e^{-2t} + c_2e^{-3t}$.

(b) Find a particular solution to the inhomogeneous equation, and combine this with your homogeneous solution to write out the general solution the inhomogeneous equation.
In this case, we're unlucky because the right hand side of our differential equation is a solution to the homogeneous equation. So we guess that the particular solution has the form $x_p = Ate^{-2t}$. Differentiating,

$$x_p' = Ae^{-2t} - 2Ate^{-2t}, \qquad x_p'' = -4Ae^{-2t} + 4Ate^{-2t}.$$

Plugging this guess in, we have

$$3e^{-2t} = x_p'' + 5x_p' + 6x_p = -4Ae^{-2t} + 4Ate^{-2t} + 5Ae^{-2t} - 10Ate^{-2t} + 6Ate^{-2t} = Ae^{-2t} \Rightarrow A = 3.$$

So the general solution to the inhomogeneous equation is

$$x(t) = 3te^{-2t} + c_1e^{-2t} + c_2e^{-3t}.$$

(c) Solve the initial value problem (for the inhomogeneous equation) with the initial conditions $x(0) = 2, x'(0) = 0$.
We have to find the coefficients bby matching the initial conditions:

$$2 = x(0) = c_1 + c_2, \qquad 0 = x'(0) = 3 - 2c_1 - 3c_2.$$

We can rearrange the second equation to read $2c_1 + 3c_2 = 3$, then subtract the first equation to get $c_2 = 1$. This implies $c_1 = 0$, so the solution to the initial value problem is

$$x(t) = 3te^{-2t} + e^{-3t}.$$

9. Consider the one-parameter family of differential equations with the parameter a:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} (e^x - 1)(a - y) \\ (e^y - 1)(a - x) \end{bmatrix}.$$

(a) Verify that the only fixed points of this system are $(0, 0)$ and (a, a).
The fized points occur when

$$0 = (e^x - 1)(a - x), \qquad 0 = (e^y - 1)(a - x),$$

and this happens precisely when $(x, y) = (0, 0)$ or $(x, y) = (a, a)$.

(b) Linearize this system about each fixed point.
If we let $f(x, y) = (e^x - 1)(a - y)$ and $g(x, y) = (e^y - 1)(a - x)$, then the coefficient matrix is

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} (a - y)e^x & 1 - e^x \\ 1 - e^y & (a - x)e^y \end{bmatrix}.$$

Evaluating at $(0, 0)$, we have

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix},$$

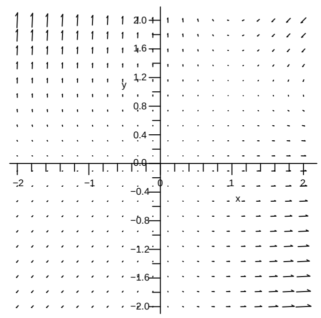
Evaluating at (a, a), we have

$$\begin{bmatrix} 0 & 1 - e^a \\ 1 - e^a & 0 \end{bmatrix}.$$

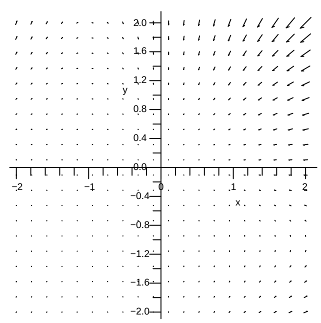
(c) What is the stability behavior of each fixed point? Explain your answer.
At the equilibrium (a, a), the eigenvalues are $\pm(1 - e^a)$, which assumes both signs. Thus (a, a) is always unstable. At the equilibrium $(0, 0)$, the eigenvalues are both a, which means that the system is stable near $(0, 0)$ if and only if $a < 0$.

(d) Sketch the phase potrait both when $a = 1$ and $a = -1$.

$a = 1$:



$a = -1$:



(e) There is one bifurcation point in a. Find it.

The bifurcation point is $a = 0$, because that is where the origin changes from being stable to being unstable.

10. Consider the damped spring equation

$$x'' + \frac{1}{100}x' + 2x = 0$$

(a) Is this underdamped, overdamped, or critically damped?

We look at the characteristic polynomial for this differential equation:

$$0 = r^2 + \frac{r}{10} + 2 \Rightarrow r = \frac{-1/100 \pm \sqrt{1/10000 - 8}}{2}.$$

This quadratic has complex roots (the discriminant is negative), and so we have the underdamped case.

(b) Sketch some representative solution curves (say with $x(0) > 0$ and $x'(0) = 0$).

11. Consider the forced spring equation

$$x'' + 9x = \cos(10t/3).$$

(a) What is the natural frequency of the spring? What is the forcing frequency?

A forced spring of this sort satisfies the equation

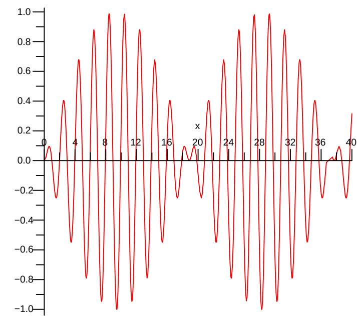
$$x'' + \omega_0^2 x = A\cos(\omega t),$$

where ω_0 is the natural frequency of the spring and ω is the forcing frequency. From this, we can read off that the natural frequency for this spring is $\sqrt{9} = 3$ and the forcing frequency is $10/3$.

(b) Find the frequency of the beats and the rapid oscillations.

One can write solutions in the form $A\sin((\omega - \omega_0)t/2)\sin((\omega + \omega_0)t/2)$. The first sin term has a low frequency, so it corresponds to the beats. Its frequency is $(1/2)(10/3 - 3) = 1/6$ (and so the period is 12π). The other term oscillates rapidly, with a frequency of $(10/3 + 3)/2 = 19/6$ (and so its period is $12\pi/19$).

(c) Sketch a representative solution curve. Be sure to label some points to indicate the scaling (e.g. the period of the beats).



12. Consider the system

$$\left[\begin{array}{c} x \\ y \end{array}\right]' = \left[\begin{array}{c} x + x^2 - xy^2 \\ y - 2xy + y^2 \end{array}\right].$$

(a) Sketch the x and y nullclines for this system.
The x nullclines satisfy the equation

$$0 = \frac{dx}{dt} = x + x^2 - xy^2 = x(1 + x - y^2).$$

So we must either have $x = 0$ or $x = y^2 - 1$. The first equation gives us the y–axis, and the second equation is a parabola opening to the right, with its vertex at $(-1, 0)$.
The y nullclines satisfy

$$0 = \frac{dy}{dt} = y - 2xy + y^2 = y(1 - 2x + y).$$

So we must have either $y = 0$ or $2x - y = 1$. The first equation gives us the x–axis, and the second equation give us a line with slope 2, and y–intercept -1.

(b) Find the equilibrium solutions.
We will find the equilibria by looking for the intersections of the nullclines. The x and y axes intersect at the origin, so $(0, 0)$ is an equilibrium. The parabola intersects the x–axis at $x = -1$, so $(-1, 0)$ is an equilibrium. The y–axis intersects the line $2x - y = 1$ at $y = -1$, so $(0, -1)$ is an equilibrium. Finally, the parabola and the line intersect at the common solutions of their equations: we plug $x = y^2 - 1$ into $2x - y = 1$ to get the quadratic $0 = 2y^2 - y - 3$, which has roots $y = -1$ and $y = 3/2$. Therefore $(0, -1)$ (which we already found) and $(5/4, 3/2)$ are the last two equilibria we're looking for.

(c) Linearize the sytem about each equilibrium.
The coefficient matrix for the linearization about (x_0, y_0) is

$$\left[\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array}\right]\Bigg|_{(x_0,y_0)} = \left[\begin{array}{cc} 1 + 2x - y^2 & -2xy \\ -2y & 1 - 2x + 2y \end{array}\right]\Bigg|_{(x_0,y_0)}$$

We evaluate this at each equilibrium. First $(0, 0)$:

$$\left[\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array}\right]\Bigg|_{(x_0,y_0)} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right],$$

then $(-1, 0)$:

$$\left[\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array}\right]\Bigg|_{(x_0,y_0)} = \left[\begin{array}{cc} -1 & 0 \\ 0 & 3 \end{array}\right],$$

then $(0, -1)$:

$$\left[\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array}\right]\Bigg|_{(x_0,y_0)} = \left[\begin{array}{cc} 0 & 0 \\ 2 & -1 \end{array}\right],$$

and finally $(5/4, 3/2)$:

$$\left[\begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array}\right]\Bigg|_{(x_0,y_0)} = \left[\begin{array}{cc} 5/4 & -5 \\ -3 & 3/2 \end{array}\right].$$

(d) Classify these equilibria as stable or unstable.

To classify the equilibria, we look at the eigenvalues of the linearization. For $(0, 0)$, the coefficient matrix of the linearization is

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right],$$

which haas $\lambda = 1$ as its only eigenvalue. Therefore, $(0, 0)$ is unstable. For $(-1, 0)$, the coefficient matrix is

$$\left[\begin{array}{cc} -1 & 0 \\ 0 & 3 \end{array}\right],$$

which has eigenvalues $\lambda = -1, 3$, so $(-1, 0)$ is also unstable. The equilibrium $(0, -1)$ has the linearized matrix

$$\left[\begin{array}{cc} 0 & 0 \\ 2 & -1 \end{array}\right],$$

which has eigenvalues $\lambda = 0, -1$. The zero eigenvalue means our stability test is inconclusive. Finally, we look at the linearization near the equilibrium $(5/4, 3/2)$:

$$\left[\begin{array}{cc} 5/4 & -5 \\ -3 & 3/2 \end{array}\right],$$

which has eigenvalues $\lambda = \frac{11/4 \pm \sqrt{121/16+105/2}}{2}$. One of these eigenvalues is positive, so $(5/4, 3/2)$ is unstable.