Solutions to the Practice Problems
Math 211
March 22, 2006

1. Consider the system

$$\begin{cases} \frac{dx_1}{dt} &= x_1 + x_2 \\ \frac{dx_2}{dt} &= x_2. \end{cases}$$

(a) Is $x_1(t) = te^t, x_2(t) = e^t$ a solution? Explain your answer.
We plug our candidate solution into the equations:

$$\begin{aligned} x_1' &= \frac{d}{dt}(te^t) = te^t + e^t = x_1 + x_2 \\ X_2' &= \frac{d}{dt}(e^t) = e^t = x_2, \end{aligned}$$

so this is a solution.

(b) Is $x_1(t) = e^{2t}, x_2(t) = e^{-t}$ a solution? Explain your answer.
Again, we plug our candidate into the equations:

$$\begin{aligned} x_1' &= \frac{d}{dt}(e^{2t}) = 2e^{2t} \neq x_1 + x_2 \\ x_2' &= \frac{d}{dt}(e^{-t}) = -e^{-t} \neq x_2, \end{aligned}$$

so this is not a solution.

2. Find the equilibrium (i.e. constant) solutions of the system

$$\begin{cases} \frac{dx_1}{dt} &= x_1 - 2x_1x_2 \\ \frac{dx_2}{dt} &= x_1x_2 - x_2. \end{cases}$$

The equilibria will satisfy $x_1' = 0, x_2' = 0$. Plugging this into the differential equations, we have

$$\begin{aligned} 0 &= x_1' = x_1 - 2x_1x_2 = x_1(1 - 2x_2) \\ 0 &= x_2' = x_2' = x_1x_2 - x_2 = x_2(x_1 - 1). \end{aligned}$$

The first equation says either $x_1 = 0$ or $x_2 = 1/2$, while the second equation says either $x_2 = 0$ or $x_1 = 1$. Thus the equilibria are $(x_1, x_2) = (1, 0)$ and $(x_1, x_2) = (0, 1/2)$.

3. Consider the second order differential equation

$$\frac{d^2x}{dt^2} + t\frac{dx}{dt} + t^2x = 0.$$

(a) Rewrite this as a 2×2 system of first order equations.
We set $x_1 = x$ and $x_2 = x'$. Then we have the system

$$\begin{aligned} x_1' &= x' = x_2 \\ x_2' &= x'' = -tx' - t^2x = -tx_1 - t^2x_2. \end{aligned}$$

(b) Verify that the only equilibrium solution is $x_1 = 0, x_2 = 0$.
Any equilibrium will satisfy $x_1' = 0$ and $x_2' = 0$, so we use the differential equations to solve for x_1 and x_2.

$$0 = x_1' = x_2 \Rightarrow x_2 = 0,$$

and so

$$0 = x_2' = -tx_1 - t^2x_2 = -tx_1 \Rightarrow x_1 = 0.$$

4. Consider the system

$$\begin{cases} \frac{dx_1}{dt} &= x_1 + x_2 \\ \frac{dx_2}{dt} &= 2x_1 - x_2 \end{cases}$$

(a) Rewrite this system in matrix form:

$$\frac{d\vec{x}}{dt} = A\vec{x},$$

where A is a 2×2 matrix.

We have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We can check this by multiplying the matrices out to get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_1 + x_2 \\ 2x_1 - x_2 \end{bmatrix}.$$

(b) Find the eigenvalues and eigenvectors of A.

The eigenvalues λ will satisfy the quadratic

$$0 = \det \begin{bmatrix} \lambda - 1 & -1 \\ -2 & \lambda + 1 \end{bmatrix} = \lambda^2 - 3,$$

and so $\lambda = \pm\sqrt{3}$.

We find the eigenvalue for $\lambda = \sqrt{3}$ first, which satisfies the equation

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sqrt{3}a \\ \sqrt{3}b \end{bmatrix} \Rightarrow b = (\sqrt{3} - 1)a,$$

so we can choose the eigenvector

$$v_+ = \begin{bmatrix} 1 \\ \sqrt{3} - 1 \end{bmatrix}.$$

(Remember, an eignevector is only determined up to scaling, so we can choose one of the components to be 1.)

For $\lambda = -\sqrt{3}$, we have

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\sqrt{3}a \\ -\sqrt{3}b \end{bmatrix} \Rightarrow b = (-\sqrt{3} - 1)a,$$

so we can choose the eigenvector

$$v_- = \begin{bmatrix} 1 \\ -\sqrt{3} - 1 \end{bmatrix}.$$

(c) Write down the general solution of the system.

We have distinct eigenvalues, so the eigenvalue/eigenvector solutions give us two distinct solutions. The general solution is a linear combination of these:

$$\vec{x}(t) = c_+ e^{\sqrt{3}t} \begin{bmatrix} 1 \\ \sqrt{3} - 1 \end{bmatrix} + c_- e^{-\sqrt{3}t} \begin{bmatrix} 1 \\ -\sqrt{3} - 1 \end{bmatrix}.$$

(d) Solve the initial value problem for this system with the initial condition $x_1(0) = 2, x_2(0) = -1$.

We have to solve

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_+ \begin{bmatrix} 1 \\ \sqrt{3} - 1 \end{bmatrix} + c_- \begin{bmatrix} 1 \\ -\sqrt{3} - 1 \end{bmatrix},$$

which we can rewrite as $2 = c_+ + c_-$ and $-1 = (\sqrt{3} - 1)c_+ + (-\sqrt{3} - 1)c_-$. The solution to this system is

$$c_+ = \frac{1 + 2\sqrt{3}}{2\sqrt{3}}, \qquad c_- = \frac{-1 + 2\sqrt{3}}{2\sqrt{3}}.$$

(e) Is the origin a stable or unstable equilibrium?

One of the eigenvalues is positive, so the origin is unstable. The phase plot looks like an unstable saddle.

(f) Sketch some typical solution curves to the system in the phase plane.

5. Consider the system

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \vec{x}.$$

(a) Find the eigenvalues and eigenvectors for the coefficient matrix A for this system.
The eigenvalues λ will satisfy

$$0 = \det \begin{bmatrix} \lambda - 2 & -1 \\ 1 & \lambda - 4 \end{bmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,$$

so the only eigenvalue is $\lambda = 3$ (it's a double root). The eigenvector associated to this eigenvalue satisfies

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix} \Rightarrow a = b,$$

so we choose the eigenvector

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Is the origin a stable or unstable equilibrium?
The origin is unstable because there is a positive eigenvalue (namely, $\lambda = 3$).

(c) Solve the inital value problem with the initial condition $x_1(0) = 1, x_2(0) = -1$.
We will look for a solution of the form

$$\vec{x}(t) = e^{3t}\vec{v_0} + te^{3t}\vec{v_1}.$$

With this form, we must have

$$\vec{v_0} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

which you can see by evaluating the solution at $t = 0$. Now we plug this into the differential equation to see what $\vec{v_1}$ has to be:

$$\begin{aligned} \frac{d\vec{x}}{dt} &= 3e^{3t}\vec{v_0} + (3te^{3t} + e^{3t})\vec{v_1} = e^{3t}(3\vec{v_0} + \vec{v_1}) + 3te^{3t}\vec{v_1} \\ A\vec{x} &= A(e^{3t}\vec{v_0} + te^{3t}\vec{v_1}) = e^{3t}A\vec{v_0} + te^{3t}A\vec{v_1}. \end{aligned}$$

Equating the terms with e^{3t} and the terms with te^{3t}, we see

$$A\vec{v_1} = 3\vec{v_1}, \qquad A\vec{v_0} = 3\vec{v_0} + \vec{v_1} \Rightarrow \vec{v_1} = A\vec{v_0} - 3\vec{v_0} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

Observe that $\vec{v_1}$ is indeed an eigenvector of A with eigenvalue 3.

(d) Write down the general solution to this system.
Let $\vec{v_0}$ be the initial condition and look for a solution of the form $\vec{x} = e^{3t}\vec{v_0} + te^{3t}\vec{v_1}$. Then, as above,

$$\frac{d\vec{x}}{dt} = 3e^{3t}\vec{v_0} + (3te^{3t} + e^{3t})\vec{v_1} = A\vec{x} = e^{3t}A\vec{v_0} + te^{3t}A\vec{v_1}.$$

We separate the terms with e^{3t} and those with te^{3t} again, to get

$$A\vec{v_1} = 3\vec{v_1}, \qquad A\vec{v_0} = 3\vec{v_0} + \vec{v_1}.$$

We rearrange the last equation, and get the general solution as

$$\vec{x}(t) = e^{3t}\vec{v_0} + te^{3t}(A\vec{v_0} - 3\vec{v_0}),$$

where $\vec{v_0}$ is the initial condition.

(e) Sketch some of the solution curves in to this system in the phase plane.

6. Consider the system

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 2 \\ -1 & a \end{bmatrix} \vec{x},$$

where a is a real parameter.

(a) Find the eigenvalues in terms of a.
The eigenvalues satisfy

$$0 = \det \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - a \end{bmatrix} = \lambda^2 - (a+1)\lambda + a + 2 \Rightarrow \lambda = \frac{(a+1) \pm \sqrt{(a+1)^2 - 4(a+2)}}{2} = \frac{-(a+1) \pm \sqrt{a^2 - 2a - 7}}{2}.$$

(b) Sketch the phase portrait of the system near the origin when $a = 0$ and when $a = -1$.

When $a = 0$, the eigenvalues are

$$\lambda = \frac{1 \pm i\sqrt{7}}{2},$$

which is a pair of complex numbers with positive real part. Thus the origin looks like an unstable sprial near the origin.

When $a = -1$, the eigenvalues are $\pm i\sqrt{3}/2$, which are pure imaginary (the real parts of both eigenvalues are zero). This means the origin is stable, but not strictly stable, and solutions near the origin trace small circles centered at the origin in the phase plane.

(c) For which values of a are there straight line solutions? (Hint: what do the solutions look like when the eigenvalues are complex numbers?)

Observe that when the eigenvalues are complex (i.e. they have nonzero imaginary part), the solutions always spiral or go in circles. In particular, there are no straight line solutions when the eigenvalues are complex. To get straight line solutions, we need real eigenvalues, which happens if and only if the discriminant $a^2 - 2a - 7$ is non zero. In other words, we need

$$a^2 - 2a - 7 \geq 0.$$

This happens only when $a \leq 1 - \sqrt{8}$ or $a \geq 1 + \sqrt{8}$.

(d) For which values of a is the origin a stable equilibrium?

Stability occurs if and only if the real part of all the eigenvaules is nonpositive, which happens precisely when $a \leq -1$.

7. Consider the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0.$$

(a) Rewrite this equation as a first order system.

If we make the substitution

$$x_1 = x, \qquad x_2 = x',$$

the corresponding system is

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_1 - 4x_2, \end{aligned}$$

which we can rewrite in matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(b) Find the general solution to this equation.

We could work with the system, finding eigenvalues and eigenvectors, but in this case it's easier to try a solution of the form $x(t) = e^{\lambda t}$. Plugging this into the equation, we get

$$0 = x'' + 4x' + 4x = e^{\lambda t}(\lambda^2 + 4\lambda + 4) = e^{\lambda t}(\lambda + 2)^2.$$

Thus we must have $\lambda = -2$ (which is a double root of the characteristic polynomial), so one solution has the form e^{-2t}. To find the other solution, we multiply by t. Thus the general solution is

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

(c) Solve the initial value problem with $x(0) = -1, x'(0) = 1$.

We have to find the coefficients c_1 and c_2. To do this, we plug in the initial conditions:

$$-1 = x(0) = c_1 \qquad 1 = x'(0) = -2c_1 + c_2,$$

so $c_1 = -1$ and $c_2 = -1$ and the solution to the initial value problem is

$$x(t) = -e^{-2t} - te^{-2t}.$$