Solutions to the Practice Problems

Math 211
Febuary 10, 2006

1. For each of the following differential equations, decide whether the given function is a solution.

   (a) $\frac{dx}{dt} = (t+1)(x^2-1)$, $x = -\frac{1+\exp(t^2+2t)}{1-\exp(t^2-2t)}$ (here $\exp(z) = e^z$)
   We have that
   $$x' = -\frac{(4t+4)e^{t^2+2t}}{(1-e^{t^2+2t})^2}, \qquad x^2 - 1 = \frac{4e^{t^2+2t}}{(1-e^{t^2+2t})^2}.$$
   Thus $x' = (t+2)(x^2-1)$ and it does satisfy the equation.

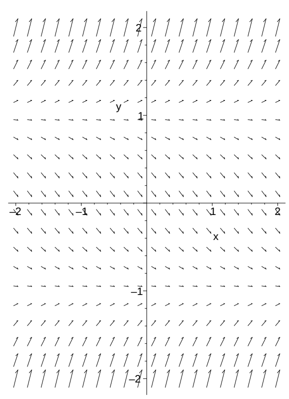
   (b) $\frac{dx}{dt} + 2xt = t$, $x = 20e^{t^2}$
   We have that $x' = -2te^{t^2}$, so
   $$x' + 2tx = -2te^{t^2} + 2t(2 - e^{t^2}) = 4t(1 - e^{t^2}) \neq t.$$
   So the function does not satisfy the differential equation.

   (c) $\frac{dx}{dt} = xt$, $x = e^{t^2/2}$
   We have that $x' = te^{t^2/2} = xt$, so it does satisfy the differential equation.

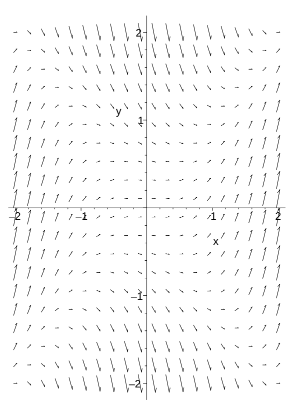
2. Sketch the slope field and some typical solution curves for each of the following differential equations.

   See me if you have questions about sketching some solution curves given the slope field. It's kind of like playing connect-the-dots.

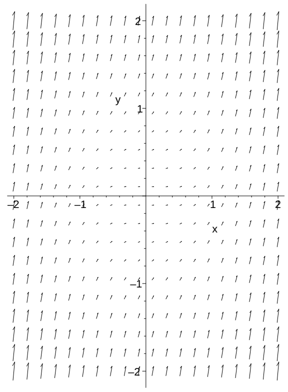
   (a) $\frac{dx}{dt} = x^2 - 1$



   (b) $\frac{dx}{dt} + x^2 - t^2 = 0$



   (c) $\frac{dx}{dt} = x^2 + t^2$



3. Solve each of the given initial value problems.

   (a) $\frac{dx}{dt} - 2xt = t$, $x(0) = 1$
   We multiply the equation by the integrating factor

$$\mu(t) = e^{\int(-2t)dt} = e^{-t^2}$$

   to get the equation

$$te^{-t^2} = \frac{dx}{dt}e^{-t^2} - 2xte^{-t^2} = \frac{d}{dt}(xe^{-t^2}).$$

   Integrating both sides of this equation with respect to t, we get

$$-\frac{1}{2}e^{-t^2} + c = \int te^{-t^2}dt = xe^{-t^2} \Rightarrow x = ce^{t^2} - \frac{1}{2}.$$

   All that remains is to determine the constant c, which we can do using the initial condition:

$$1 = x(0) = c - \frac{1}{2} \Rightarrow c = \frac{3}{2} \Rightarrow x(t) = \frac{3}{2}e^{t^2} - \frac{1}{2}.$$

   (b) $\frac{dx}{dt} - t^2e^{-x} = 0$, $x(0) = 4$
   For this equation we separate variables:

$$e^x\frac{dx}{dt} = t^2.$$

   Integrating both sides of the equation with respect to t, we get

$$\frac{1}{3}t^3 + c = \int t^2dt = \int e^x\frac{dx}{dt}dx = \int e^x dx = e^x,$$

   and so

$$x = \ln(t^3/3 + c).$$

   To determine c we use the initial condition:

$$4 = x(0) = \ln(c) \Rightarrow c = e^4 \Rightarrow x = \ln(t^3/3 + e^4).$$

   (c) $\frac{dx}{dt} - xe^x = 0$, $x(0) = 0$
   Notice this equation has the form $x' = f(x) = xe^x$. Also, the initial condition lies in the zero set of f: $f(0) = 0$. Thus the solution is the constant solution $x(t) \equiv 0$.

   (d) $\frac{dx}{dt} = x^2$, $x(0) = 1$
   We can separate variables:

$$\frac{1}{x^2}\frac{dx}{dt} = 1 \Rightarrow t + c = \int\frac{dx}{x^2} = -\frac{1}{x} \Rightarrow x = -\frac{1}{t+c}.$$

   From the intial condition

$$1 = x(0) = -\frac{1}{c} \Rightarrow c = -1 \Rightarrow x = -\frac{1}{t-1} = \frac{1}{1-t}.$$

(e) $\frac{dx}{dt} + x/t = t$, $x(1) = 1$
We multiply by the integrating factor

$$\mu(t) = e^{\int dt/t} = e^{\ln t} = t$$

to get

$$t^2 = t\frac{dx}{dt} + x = \frac{d}{dt}(tx) \Rightarrow \frac{1}{3}t^3 + c = tx \Rightarrow x = \frac{1}{3}t^2 + \frac{c}{t}.$$

From the initial condition we get

$$1 = x(1) = \frac{1}{3} + c \Rightarrow c = \frac{2}{3} \Rightarrow x = \frac{1}{3}t^2 + \frac{2}{3t}.$$

4. Consider the differential equation

$$\frac{dx}{dt} = x(x^2 - 1).$$

(a) Find all the equilibria (constant solutions) of this equation.
The constant solutions satisfy

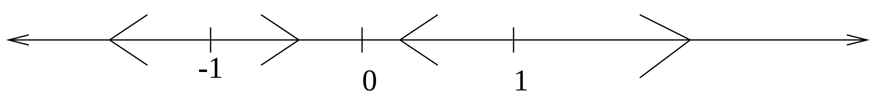
$$0 = \frac{dx}{dt} = x(x^2 - 1) = x(x-1)(x+1).$$

Therefore, we must have $x = 0, 1, -1$.

(b) Classify each of these equilibria as sinks, sources, or nodes
We linearize the equation an apply the classification test. We have $x' = f(x) = x(x^2 - 1)$, so $\frac{df}{dx} = 3x^2 - 1$. Evaluate this derivative at each equilibrium:

$$\frac{df}{dx}(0) = -1 < 0, \qquad \frac{df}{dx}(1) = 2 > 0, \qquad \frac{df}{dx}(-1) = 2 > 0,$$

so $x = 0$ is a sink and $x = \pm 1$ are both sources.



(c) Sketch some typical solution curves.

5. Consider the differential equation

$$\frac{dx}{dt} = ax - x^2.$$

(a) Draw the slope field and sketch some representative solution curves for the parameter values $a = 1, 0, -1$.

(b) Find the equilibrium solutions for the parameter values $a = 1, 0, -1$.
Again, the equilibirum solutions occur when $0 = ax - x^2 = x(a - x)$, or when $x = 0, a$. Thus, when $a = 1$ the equilibria are $x = 0, 1$, when $a = 0$ the only equilibrium is $x = 0$, and when $a = -1$ the equlibria are $x = 0, -1$.

(c) Classify these equilibria as sinks, sources, or nodes.
We linearize the equation in each case. First, for $a = 1$ we have

$$\left.\frac{d}{dx}\right|_{x=0}(x - x^2) = 1, \qquad \left.\frac{d}{dx}\right|_{x=0}(x - x^2) = -1,$$

so $x = 0$ is a source and $x = 1$ is a sink. For $a = -1$ we have

$$\left.\frac{d}{dx}\right|_{x=0}(-x - x^2) = -1, \qquad \left.\frac{d}{dx}\right|_{x=0}(-x - x^2) = 1,$$

so $x = 0$ is a sink and $x = -1$ is a source. Finally, for $a = 0$ we have

$$\left.\frac{d}{dx}\right|_{x=0}(-x^2) = 0,$$

so the test is inconclusive. However, we see that $-x^2 < 0$ for $x \neq 0$, which means that the solutions are decreasing so long as they start with a nonzero initial value. Thus solutions starting above the $x = 0$ equilibrium go towards from it and solutions starting below the $x = 0$ equilibrium go away from it, so $x = 0$ is a node.

(d) Does this differential equation have a bifurcation point? Find the bifurcation point if it exists. Be sure to explain your aswer.

Yes, $a = 0$ is the bifurcation point because the differential equation has one equilibrium there, while it has two equilibria (a sink and a source) for other values of a.

6. (a) Write down a differential equation with two equilibria, one of which is a sink and one of which is a source. (Be sure to justify your answer.)

Consider the differential equation

$$\frac{dx}{dt} = 1 - x^2 = f(x).$$

The equilibria are the zeroes of f, which are $x = \pm 1$. To see whether these equilibria are sinks or soruces, we evaluate the derivative of f:

$$\frac{df}{dx} = -2x, \qquad \frac{df}{dx}(1) = -2, \qquad \frac{df}{dx}(-1) = 2.$$

Thus $x = 1$ is a sink and $x = -1$ is a source.

(b) Write down a differential equation with equilibrium, which is a node. (Be sure to justify your answer.)

Consider the differential equation

$$\frac{dx}{dt} = x^2 = f(x).$$

The equilibrium is the only zero of the function f, which is $x = 0$. Moreover, for $x \neq 0$ we have $\frac{dx}{dt} = x^2 > 0$, so all other solutions are increasing. Thus, solutions with a positive initial value go away from the $x = 0$ equilibrium, while solutions with a negative initial value go towards the $x = 0$ equilibrium. In other words, $x = 0$ is a node.

7. Consider the differenital equation

$$\frac{dx}{dt} = x(x^2 - 1) - a,$$

where a is a parameter.

(a) Does this system have a bifurcation point? If it does, find the bifucation point.

Yes, $a = \pm 2/(3\sqrt{3}$ are bifurcation points. Consider graph of $f(x) = x(x^2-1)$; this is a cubic which crosses the x axis at $x = 0, \pm 1$ (these are the equilibria when $a = 0$), with a local maximum of height $2/(3\sqrt{3})$ and a local minimum of $-2/(3\sqrt{3})$. Subtracting a from $f(x)$ has the effect of translating the graph down by a, shifting the points where it crosses the the x axis. When $a = \pm 2(3\sqrt{3})$, the local maximum/minimum touches the axis. Thus, for $a < -2/(3\sqrt{3})$ or $a > 2/(3/sqrt3)$ the graph crosses the axis only, giving you one equilibrium. For $-2/(3\sqrt{3}) < a < 2/(3\sqrt{3})$ we have three equilibria, while for $a = \pm 2/(3\sqrt{3})$ we have two equilibria.

(b) Describe how the behavior of the system depends on a.

We break this into cases.

For $-2(3\sqrt{3}) < a < 2/(3\sqrt{3})$ we have three equilibria $x_1 < x_2 < x_3$. The function $x^3 - x - a$ is positive for $x_1 < x < x_2$ and for $x > x_3$, while it's negative for $x < x_1$ and $x_2 < x < x_3$. Thus x_1 and x_3 are sources while x_2 is a sink.

For $a > 2/(3\sqrt{3})$ or $a < -2/(3\sqrt{3})$ we have one equilibrium x_1. The function $x^3 - x - a$ is positive to the right of x_1 and negative to the left, so x_1 is a source.

For $a = 2/(3\sqrt{3})$ we have two equilibria $x_1 < x_2$. The function $x^3 - x - a$ is negative for $x < x_1$ and $x_1 < x < x_2$, while it's positive for $x > x_2$. Thus x_1 is a node and x_2 is a source.

For $a = -2/(3\sqrt{3})$ we have two equilibria $x_1 < x_2$. The function $x^3 - x - a$ is negative for $x < x_1$, while it's positive for $x_1 < x < x_2$ and $x > x_2$. Thus x_1 is a source and x_2 is a node.

(c) Sketch some typical solution curves, depending on the value of the parameter a.