Solutions to the Midterm Exam

Math 211

April 4, 2006

1. Consider the system

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{x_1}{t} - tx_2 \\ \frac{dx_2}{dt} &= \frac{x_1}{t}.\end{aligned}$$

   (a) (3 points) Is $x_1(t) = t^2, x_2(t) = t$ a solution? Explain your answer.
   We plug in $x_1 = t^2$ and $x_2 = t$. Then $x_1' = 2t$, but

$$\frac{x_1}{t} - tx_2 = t - t^2 \neq 2t,$$

   so this is not a solution.

   (b) (3 points) Is $x_1(t) = t\cos(t), x_2(t) = \sin(t)$ a solution? Explain your answer.
   Again, we plug in $x_1 = t\cos t$ and $x_2 = \sin t$. Then $x_1' = \cos t - t\sin t$ and

$$\frac{x_1}{t} - tx_2 = \cos t - t\sin t.$$

   Also, $x_2' = \cos t = x_1/t$. Putting these two together, we see that this is a solution.

2. (4 points) Verify that the only equilibria of the system

$$\begin{aligned}\frac{dx_1}{dt} &= 20x_1 - 10x_1x_2 \\ \frac{dx_2}{dt} &= -5x_2 + x_1x_2\end{aligned}$$

   are $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (5, 2)$.
   The equilibria will satisfy $x_1' = 0$ and $x_2' = 0$. Plugging this into the equations, we have the system

$$\begin{aligned}0 &= 20x_1 - 10x_1x_2 \\ 0 &= -5x_2 + x_1x_2\end{aligned}$$

   We can factor the first equation as $0 = x_1(20 - 10x_2)$, which says that either $x_1 = 0$ or $x_2 = 2$. If $x_1 = 0$, the second equation reads $0 = -5x_2$, which implies $x_2 = 0$. On the other hand, if $x_2 = 2$, the second equation reads $0 = -10 + 2x_1$, which implies $x_1 = 5$. Thus the only equilibria are $(0, 0)$ and $(5, 2)$.

3. Consider the system

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \vec{x} = Ax.$$

   (a) (5 points) Find the eigenvalues of A.
   The eigenvalues satisfy

$$0 = \det \begin{bmatrix} \lambda - 1 & -2 \\ -1 & \lambda + 3 \end{bmatrix} = \lambda^3 + 2\lambda - 5,$$

   so

$$\lambda = \frac{-2 \pm \sqrt{4 + 20}}{2} = -1 \pm \sqrt{6}.$$

   (b) (5 points) Find the eigenvector of A and write down the general solution.
   We first find the eigenvector for $\lambda = -1 + \sqrt{6}$:

$$\begin{bmatrix} (-1 + \sqrt{6})a \\ (-1 + \sqrt{6})b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow b = (-1 + \sqrt{3/2})a,$$

   so the eigenvector is (up to scaling)

$$\vec{v_1} = \begin{bmatrix} 1 \\ -1 + \sqrt{3/2} \end{bmatrix}.$$

Next we find the eigenvector for $\lambda = -1 - \sqrt{6}$:

$$\begin{bmatrix} (-1-\sqrt{6})a \\ (-1-\sqrt{6})b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow b = (-1-\sqrt{3/2})a,$$

so the eigenvector is (up to scaling)

$$\vec{v_2} = \begin{bmatrix} 1 \\ -1-\sqrt{3/2} \end{bmatrix}.$$

Finally, the general solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{-\lambda_2 t}\vec{v_2} = c_1 e^{(-1+\sqrt{6})t} \begin{bmatrix} 1 \\ -1+\sqrt{3/2} \end{bmatrix} + c_2 e^{(-1-\sqrt{6})t} \begin{bmatrix} 1 \\ -1-\sqrt{3/2} \end{bmatrix}.$$

(c) (5 points) Sketch some solution curves in the phase plane near the origin and characterize the origin $\vec{0}$ as an unstable, stable, or strictly stable equilibrium. Be sure to explain your answer.
There is one positive and one negative eigenvalue, which tells us that the origin is a saddle. In particular, $\vec{0}$ is an unstable equilibrium.

(d) (5 points) Solve the initial value problem with the initial condition

$$\vec{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

We know the form of the general solution, and have to find constants c_1 and c_2 to match the initial conditions. We have

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} 1 \\ -1+\sqrt{3/2} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1-\sqrt{3/2} \end{bmatrix},$$

which givves the system

$$\begin{aligned} -1 &= c_1 + c_2 \\ 2 &= (-1+\sqrt{3/2})c_1 + (-1-\sqrt{3/2})c_2. \end{aligned}$$

This system has the solution $c_1 = -1/2 + 1/\sqrt{6}$, $c_2 = -1/2 - 1/\sqrt{6}$, so we have

$$x(t) = \left(-\frac{1}{2} + \frac{1}{\sqrt{6}}\right) e^{(-1+\sqrt{6})t} \begin{bmatrix} 1 \\ -1+\sqrt{3/2} \end{bmatrix} + \left(-\frac{1}{2} - \frac{1}{\sqrt{6}}\right) e^{(-1-\sqrt{6})t} \begin{bmatrix} 1 \\ -1-\sqrt{3/2} \end{bmatrix}.$$

4. Consider the differential equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0.$$

(a) (5 points) What is the general solution to this differential equation?
We try solutions of the form $x(t) = e^{st}$. Plugging this in, we find

$$0 = x'' + 6x' + 9x = e^{st}(s^2 + 6s + 9) = (s+3)^2 e^{st} \Rightarrow s = -3.$$

Notice we have a double root, so the general solution is

$$x(t) = c_1 e^{-3t} + c_2 t e^{-3t}.$$

(b) (5 points) Solve the initial value problem with the initial condition $x(0) = -2, x'(0) = 1$.
We plug in our initial conditions:

$$-2 = x(0) = c_1, \qquad 1 = x'(0) = -3c_1 + c_2 = 6 + c_2 \Rightarrow c_2 = -5,$$

so

$$x(t) = -2e^{-3t} - 5te^{-3t}.$$

5. Consider the system

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 2 \\ a & 1 \end{bmatrix} \vec{x} = A\vec{x},$$

where a is a parameter which can be any real number.

(a) (5 points) Find the eigenvalues of A in terms of a.

The eigenvalues satisfy

$$0 = \det \begin{bmatrix} \lambda - 1 & -2 \\ -a & \lambda - 1 \end{bmatrix} = \lambda^2 - 2\lambda + 1 - 2a,$$

so

$$\lambda = \frac{2 \pm \sqrt{4 - 4(1 - 2a)}}{2} = 1 \pm \sqrt{2a}.$$

(b) (5 points) When does this system have straight line solutions? (Hint: think about what the solution curves look like, depending on whether the eigenvalues are real or complex numbers.)

There are straight line solutions precisely when the eigenvalues are real numbers, which happens when $a \geq 0$.