Solutions to the First Midterm Exam

Math 211

Feb. 17, 2005

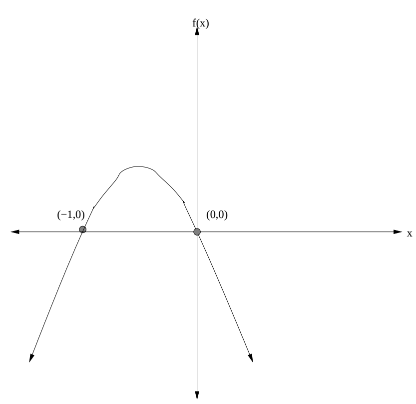
1. Consider the differential equation
$$\frac{dx}{dt} = tx^2 + \frac{1}{1-t}.$$

   (a) (3 points) Is $x(t) = \frac{1}{1-t}$ a solution? Be sure to explain your answer.
   We first compute the left hand side of the equation:
   $$\frac{dx}{dt} = \frac{d}{dt}\frac{1}{1-t} = \frac{1}{(1-t)^2}.$$
   Next we compute the right hand side:
   $$tx^2 + \frac{1}{1-t} = \frac{t}{(1-t)^2} + \frac{1}{1-t} = \frac{t}{(1-t)^2} + \frac{1-t}{(1-t)^2} = \frac{1}{(1-t)^2}.$$
   The two match, so $x(t) = 1/(1-t)$ is a solution.

   (b) (2 points) Is $x(t) = t^2$ a solution? Be sure to explain your answer.
   We first compute the left hand side of the equation:
   $$\frac{dx}{dt} = \frac{d}{dt}(x^2) = 2t.$$
   next we compute the right hand side:
   $$tx^2 + \frac{1}{1-t} = t^5 + \frac{1}{1-t} = \frac{-t^6 + t^5 + 1}{1-t}.$$
   The two don't match, so $x = t^2$ is not a solution.

2. (10 points) Solve the initial value problem
$$\frac{dx}{dt} = te^x, \qquad x(0) = 1.$$
   This is a separable equation:
$$\frac{dx}{dt} = te^x \Leftrightarrow e^{-x}\frac{dx}{dt} = t.$$
   Integrating both sides of the equation with respect to t, we get
$$\frac{t^2}{2} + c = \int e^{-x}\frac{dx}{dt}dt = \int e^{-x}dx = -e^{-x}.$$
   Next we solve for x:
$$-e^{-x} = \frac{t^2}{2} + c \Leftrightarrow e^{-x} = -\frac{t^2}{2} + c \Leftrightarrow x = -\ln[c - \frac{t^2}{2}].$$
   We find c using the initial condition:
$$1 = x(0) = -\ln[c] \Rightarrow c = \frac{1}{e} \Rightarrow x = -\ln[\frac{1}{e} - \frac{t^2}{2}].$$

3. Consider the differential equation $\frac{dx}{dt} = f(x)$, where the graph of f is drawn below.



(a) (5 points) Explain why the only two equilibria are $x = 0$ and $x = -1$.
The equilibria occur when

$$0 = \frac{dx}{dt} = f(x),$$

and (by the graph) the only zeroes of f are $x = 0$ and $x = -1$.

(b) (5 points) Suppose $x(t)$ is a solution to the differential equation with $x(0) = -1/2$. What can you say about $x(t)$ as $t \to \infty$ or $t \to -\infty$?
First of all, we know that $x(0)$ lies between the two equilibria, so $x(t)$ must stay between the two equilibria for all time: $-1 < x(t) < 0$ for all t. In particular, the solutions exist for all time. Furthermore, for $-1 < x < 0$ we see that $f(x) > 0$. This means, because $dx/dt = f(x) > 0$, that $x(t)$ is a monotone increasing function. Thus $x(t) \to 0$ as $t \to \infty$ and $x(t) \to -1$ as $t \to -\infty$.

4. (10 points) Solve the initial value problem

$$\frac{dx}{dt} = \frac{x}{t} + 4t^2, \qquad x(1) = 1.$$

This is a first order linear equation, with $g(t) = 1/t$ and $r(t) = 4t^2$. The integrating factor is

$$\mu(t) = e^{-\int g(t)dt} = e^{-\int dt/t} = e^{-\ln t} = e^{\ln(1/t)} = \frac{1}{t}.$$

Multiplying by μ, the equation becomes

$$4t = \frac{1}{t} \cdot \frac{dx}{dt} - \frac{x}{t^2} = \frac{d}{dt}(\frac{x}{t}).$$

Inegrating both sides of the equation with respect to t we get

$$2t^2 + c = \frac{x}{t} \Rightarrow x = 2t^3 + ct.$$

We find the constant c using the initial condition:

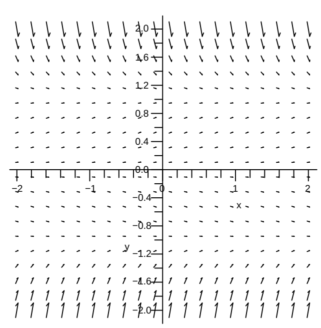
$$1 = x(1) = 2 + c \Rightarrow c = -1 \Rightarrow x = 2t^3 - t.$$

5. Consider the differential equation

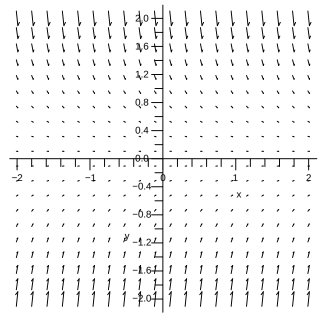
$$\frac{dx}{dt} = ax - x^3,$$

where a is a parameter.

(a) (4 points) Sketch the slope field and some representative solution curves when $a = 1$ and $a = -1$.
Here is the slope field for $a = 1$:



Here is the slope field for $a = -1$:



(b) (4 points) When $a = 1$, show that the only equilibrium (i.e. constant) solutions are $x = 0, x = 1, x = -1$. What happens when $a = -1$?

We treat the $a = 1$ case first. We're looking for the constant solutions, which occur when

$$0 = \frac{dx}{dt} = x - x^3 = x(1-x)(1+x) \Leftrightarrow x = 0, 1, -1.$$

Thus the equilibria are $x = 0, 1, -1$. We do the same analysis in the $a = -1$ case. This time we have

$$0 = \frac{dx}{dt} = -x - x^3 = -x(1+x^2) \Leftrightarrow x = 0.$$

(c) (4 points) Classify the $a = 1$ equilibria as sinks, sources, or nodes.

Recall we have $dx/dt = f(x) = x - x^3$. We can classify the equilibria after taking a derivative of f:

$$f'(x) = \frac{df}{dx} = 1 - 3x^2.$$

Evaluating at each of the equilibria, we see $f'(0) = 1 > 0$, so $x = 0$ is a source and $f(\pm 1) = -2 < 0$, so $x = 1, -1$ are sinks. Here is a picture of the phase line:

(d) (3 points) This differential equation has one bifurcation point; find it.

The bifurcation point is $a = 0$, which we can see by counting the equilibria. We have equilibria when

$$0 = \frac{dx}{dt} = ax - x^3 = x(a - x^2).$$

When $a > 0$ this cubic has three roots: $x = 0, \pm\sqrt{a}$, but when $a \leq 0$ the cubic only has one (real) root, at $x = 0$. Because the number of equilibria changes at $a = 0$, it is the bifurcation point.