

Practice Problems
Math 210
November 27, 2006

These problems are not in any particular order. The exam will be shorter. The last 8 problems form the final exam from the last time I taught this course.

- For each of the following pairs of vectors, compute $\vec{u} \cdot \vec{v}$, $\vec{u} \times \vec{v}$ and the orthogonal projection of \vec{u} onto \vec{v} .
 - $\vec{u} = (1, 0, 1)$, $\vec{v} = (1, 1, 0)$
 - $\vec{u} = (0, 1, 0)$, $\vec{v} = (0, 1, -1)$
- Let Π_1 be the plane through $(1, 1, 1)$ with normal vector $\vec{n}_1 = (1, -1, 0)$, and let Π_2 be the plane defined by $x + y + z = 1$.
 - Write down a linear equation for Π_1 .
 - Find the cosine angle between Π_1 and Π_2 .
 - Parameterize the line l of intersection between Π_1 and Π_2 .
 - Find the distance between Π_2 and $(1, 1, 1)$.
- Consider the curve $c(t) = (\cos(2t), \sin(t))$ for $0 \leq t \leq 2\pi$.
 - Sketch this curve.
 - Write down the tangent line to c at the point $(-1, 1)$. (This point corresponds to the parameter value $t = \pi/2$).
 - Is the tangent line to this curve ever parallel to the line $y = -x$? Be sure to explain your answer.
 - Set up, but do not evaluate, the integral to compute the arclength of this curve.

- Consider the function

$$f(x, y) = \int_y^{e^x} \sqrt{1+t^2} dt.$$

- Compute the partial derivatives of f .
 - Does f have any critical points? Be sure to explain your answer.
- Consider the function $f(x, y) = x^2y - xy^3$.
 - Does f have an upper or lower bound? Explain your answer.
 - Compute the partial derivatives of f .
 - Compute the directional derivative of f in the $(1/\sqrt{2}, -1/\sqrt{2})$ direction, at the point $(1, 1)$.
 - Find the direction of steepest ascent for f , starting at $(1, 1)$. Make sure to write down a unit vector.
 - Notice that $f(1, 1) = 0$. Write down the equation of the tangent line to the $f = 0$ level set, at the point $(1, 1)$.

- Consider the function

$$f(x, y) = e^{xy^3 - x^2}.$$

- Find and classify all the critical points of f .
 - Find the absolute minimum of f restricted to the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.
- Recall that two tangent directions to the graph of f are given by the vectors

$$\left(1, 0, \frac{\partial f}{\partial x}\right), \quad \left(0, 1, \frac{\partial f}{\partial y}\right).$$

- Compute a normal vector for the graph.
 - Can the tangent plane of the graph ever be parallel to the $x - z$ plane? Explain your answer.
- Find the absolute maximum and minimum of the function $f = xy$ on the ellipse $4x^2 + y^2 = 4$.
 - Evaluate $\int_D \sqrt{4 - x^2 - y^2} dA$ where $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y > 0\}$.

10. Set up, but do not evaluate $\int_D xe^{x^2y}dA$, where D is the region bounded by the curves $y = x^3$ and $y = x$. (Be careful of signs.)
11. Evaluate $\int \int_D \sqrt{1-x^2}dA$ where D is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.
12. Consider the domain $D := \{(x,y) \mid |x+y| \leq 1\}$.
- Sketch D .
 - If we change variables by $u = \frac{1}{\sqrt{2}}(x+y)$, $v = \frac{1}{\sqrt{2}}(-x+y)$, what is D in the u, v coordinate system?
 - Set up, but do not evaluate, the integral $\int \int_D \cos(\pi x - \pi y)dA$ in the (u, v) coordinate system.
13. Consider the vector field $\vec{F} = (-y, x)$.
- Sketch \vec{F} .
 - Compute $\int_\gamma \vec{F} \cdot d\vec{s}$ where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$.
14. Consider the vector field $\vec{F} = (2xe^{x^2+y^2}, 2ye^{x^2+y^2} + \cos y)$.
- Show that $\vec{F} = \nabla f$ for some f .
 - Find a function f such that $\vec{F} = \nabla f$.
 - Compute $\int_\gamma \vec{F} \cdot d\vec{s}$, where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi/2$. (Hint: you don't need to actually do an integral.)
15. Consider $\vec{F} = (-y + x^2 - y \cos(xy), x - y^3 - x \cos(xy))$.
- Is $\vec{F} = \nabla f$ for some f ? Be sure to explain your answer.
 - Compute $\int_\gamma \vec{F} \cdot d\vec{s}$. (Hint: don't actually compute the line integral.)
16. Consider the vector field $\vec{F} = (z + x^3 - yze^{xyz}, y - xze^{xyz}, -x + z^2 - xye^{xyz})$.
- Compute $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$.
 - Is $\vec{F} = \nabla f$ for some f ? Be sure to explain your answer.
 - Compute $\int \int_\Sigma \nabla \times \vec{F} \cdot \vec{n}dA$, where Σ is the upper unit hemisphere, centered at $(0,0,0)$, with the outward unit normal.
17. Consider the surface $\vec{r}(u, v) = (u \cos(v), u \sin(v), v)$.
- Compute the tangent vectors $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$.
 - Verify that this is a good parameterization, by checking that $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$ are never parallel.
 - Find the equation of the tangent plane to \vec{r} for the parameter values $u = 1, v = \pi$.
 - Is the tangent plane ever parallel to the $x - y$ plane? Be sure to explain your answer.
 - What is this surface? Can you draw a sketch of it? (Hint: fix a value of u , for instance $u = 1$ or $u = 0$, and draw the resulting curve.)
18. Compute $\int \int_\Sigma \nabla \times \vec{F} \cdot \vec{n}dA$, where $\vec{F} = (-y, x, 0)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal.
19. Compute $\int \int_\Sigma \vec{F} \cdot \vec{n}dA$, where $\vec{F} = (x + yz - \cos y, y - e^{xz} + z^2, z - x \cos(x^2y))$ and Σ is the unit sphere (centered at the origin) with the outward unit normal.
20. Compute $\int \int_\Sigma \vec{F} \cdot \vec{n}dA$, where $\vec{F} = (x + ze^{y^2+z}, y - z \cos(x+z^2), z)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal. (Hint: what is \vec{F} restricted to the plane $z = 0$?)

21. Let Π_1 be the plane through $(1, 0, 1)$ with normal $\vec{n}_1 = (1, -1, 0)$, and let Π_2 be the plane given by $x = y + z$.
- (3 points) Find $\cos \theta$, where θ is the angle between Π_1 and Π_2 .
 - (4 points) Find a linear equation for the plane Π_1 .
 - (4 points) Parameterize the line l of intersection between Π_1 and Π_2 .
 - (4 points) Find the distance between $(1, 1, 2)$ and Π_2 .

22. Consider $f(x, y) = x^2y - xy - x^2 - 6y - x + 6$.

- (5 points) Verify that the only critical points of f are $(3, 7/5)$ and $(-2, 3/5)$.
- (5 points) Classify these critical points as local maxima, local minima, saddle points, or none of the above.
- (5 points) Observe that $f(1, 1) = -2$. Write down an equation for the tangent line to the level set $\{f = -2\}$, at the point $(1, 1)$.

23. Recall that two tangent directions to the graph of a function $f(x, y)$ are

$$\left(1, 0, \frac{\partial f}{\partial x}\right), \quad \left(0, 1, \frac{\partial f}{\partial y}\right).$$

- (5 points) Write down a normal vector to the tangent plane of the graph of f .
 - (5 points) Is it ever possible that the tangent plane to the graph of f is parallel to the plane $x + y = 0$? Be sure to explain your answer.
24. Consider the composition $g(t) = f(x(t), y(t))$, where $x(1) = 1$, $x'(1) = -1$, $y(1) = 2$, $y'(1) = 3$, and $\nabla f(1, 2) = (3, -2)$.
- (3 points) What is $g'(1)$?
 - (4 points) Suppose $x'(2) = 0$ and $y'(2) = 0$. Is it necessarily true that $g'(2) = 0$? Explain your answer.
 - (3 points) Suppose $g'(-1) = 0$. Is it necessarily true that $x'(-1) = 0$ and $y'(-1) = 0$? Explain your answer.

25. (a) (5 points) Evaluate $\int \int_D e^{x^2+y^2} dA$, where D is the domain $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, x \leq 0\}$.
- (b) (5 points) Set up, but **do not evaluate** the integral $\int \int_D e^{x^2y} \sin(y^3 + x) dA$, where D is the region bounded by the line $y = 1 + x$ and the curve $y = x^2$. (It might help to draw a picture.)

26. Consider the vector field $\vec{F} = (y \cos(xy), x \cos(xy) + y^2)$ in the plane.

- (5 points) Verify that \vec{F} is the gradient of a function.
- (5 points) Find a function f such that $\vec{F} = \nabla f$.
- (5 points) Evaluate $\int_\gamma \vec{F} \cdot d\vec{s}$ where $\gamma(t) = (t, 1 - t^2)$ for $0 \leq t \leq 1$. (Hint: you do not need to compute an integral.)

27. Consider the vector field $\vec{F} = (y + yze^{xyz}, -x + xze^{xyz}, xye^{xyz})$ in three-space.

- (5 points) Compute the curl of \vec{F} .
- (5 points) Is \vec{F} the gradient of a function? Be sure to explain your answer.
- (5 points) Evaluate $\int \int_\Sigma \nabla \times \vec{F} \cdot \vec{n} dA$, where Σ is the hemisphere $\{(x, y, z) \mid x^2 + y^2 + z^2, x \geq 0\}$, oriented with the normal \vec{n} pointing away from the origin. (Hint: you have to compute an integral, but not necessarily over Σ .)

28. Consider the vector field $\vec{F} = (\cos(y^2z^{1/3}) - x + \ln y, y - e^{xz} + z, z + \sin(x^2y^6) \ln(\sqrt{1 + x^2 + y^2}))$ in three-space.

- (5 points) Compute the divergence of \vec{F} .
- (5 points) Evaluate $\int \int_\Sigma \vec{F} \cdot \vec{n} dA$, where Σ is the unit sphere, centered at the origin, oriented with the outward normal \vec{n} . (Hint: think before you compute.)

29. Consider the curve $c(t) = (\cos t, \sin t, t)$.

- (4 points) Find the equation of the tangent line to c at time $t = \pi$.
- (3 points) Is the tangent line to c ever parallel to the xy -plane? Be sure to explain your answer.

(c) (3 points) Find the length of the section of c corresponding to $0 \leq t \leq 2\pi$.

30. Consider the function $f(x, y) = x^3 - x^2y + 3xy$.

(a) (5 points) Verify that the only critical points of this function are $(0, 0)$, $(3, 9)$.

(b) (5 points) Classify these critical points as local minima, local maxima, saddle points, or none of the above.

(c) (5 points) Observe that $f(1, 1) = 3$. Write down an equation for the tangent line to the $f = 3$ level set of f at $(1, 1)$.

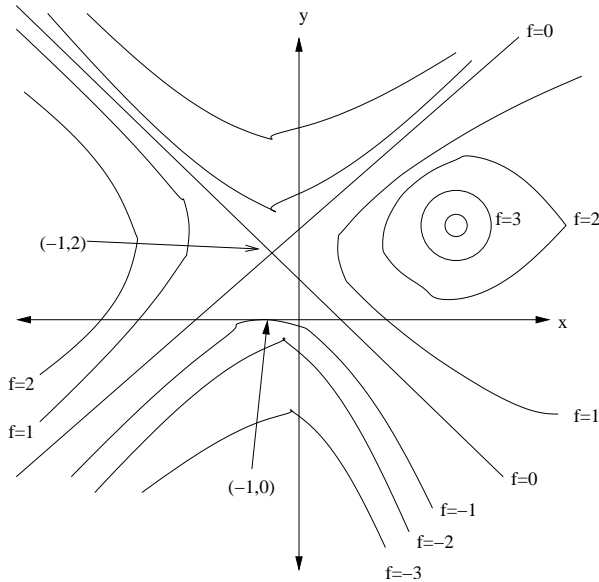
31. Consider the composition $g(t) = f(x(t), y(t))$, where $x(1) = 1$, $y(1) = 2$, $x'(1) = -1$, $y'(1) = 1$ and $\nabla f(1, 2) = (3, 2)$.

(a) (4 points) What is $g'(1)$?

(b) (3 points) Suppose $x'(2) = 0$ and $y'(2) = 0$. Is it necessarily true that $g'(2) = 0$? Be sure to explain your answer.

(c) (3 points) Suppose $g'(0) = 0$. Is it necessarily true that $x'(0) = 0$ and $y'(0) = 0$? Be sure to explain your answer.

32. Below is a sketch of some of the level curves of a function $f = f(x, y)$.



(a) (5 points) In which direction does $\nabla f(-1, 0)$ point? Be sure to explain your answer.

(b) (5 points) If $(-1, 2)$ is a critical point of f , what kind of critical point do you expect it to be? Be sure to explain your answer.

33. (5 points) Set up, but **do not evaluate**, the integral $\int \int_D x \ln(1 + x^2 + y^2) dA$ as an iterated integral, where D is the domain bounded by $x = y^2 - 2y$ and $y = x - 1$. (It might help to draw a picture.)

34. (10 points) Evaluate $\int \int_D \sqrt{x^2 + y^2 - 1} dA$, where $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4\}$.

35. Consider the vector field $\vec{F} = (ye^{xy} + 2x, xe^{xy})$ in the plane.

(a) (5 points) Verify that $\vec{F} = \nabla f$ for some function $f = f(x, y)$.

(b) (5 points) Find a function f such that $\vec{F} = \nabla f$.

(c) (5 points) Evaluate $\int_\gamma \vec{F} \cdot d\vec{s}$, where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi/2$. (Think before you compute.)

36. Consider the vector field $\vec{F} = (z \ln(1 + y^2) - x, 2y - xe^{xz}, -1 + xy \sin(1 + x^2 + y^2))$ in three-space.

(a) (5 points) Compute the divergence of \vec{F} .

(b) (5 points) Evaluate $\int \int_{\Sigma} \vec{F} \cdot \vec{n} dA$, where Σ is the unit sphere centered at the origin, oriented with the **inward** normal. (Think before you compute.)

37. Consider the vector field $\vec{F} = (yz \cos(xyz), xz \cos(xyz), y + xy \cos(xyz))$ in three-space.

(a) (5 points) Compute the curl of \vec{F} .

(b) (5 points) Is \vec{F} the gradient of a function $f(x, y, z)$? Be sure to explain your answer.

(c) (5 points) Evaluate $\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA$, where Σ is the upper unit hemisphere centered at the origin parameterized by $\{(\sqrt{1-u^2} \cos v, \sqrt{1-u^2} \sin v, u) \mid 0 \leq u \leq 1, 0 \leq v \leq 2\pi\}$, oriented with the **upward** normal. (Hint: you don't have to integrate over Σ , you can replace it with another surface which has the same boundary.)