

Practice Problems  
Math 210  
December 4, 2006

1. For each of the following pairs of vectors, compute  $\vec{u} \cdot \vec{v}$ ,  $\vec{u} \times \vec{v}$  and the orthogonal projection of  $\vec{u}$  onto  $\vec{v}$ .

(a)  $\vec{u} = (1, 0, 1), \vec{v} = (1, 1, 0)$

$$\vec{u} \cdot \vec{v} = (1, 0, 1) \cdot (1, 1, 0) = 1 + 0 + 0 = 1$$

$$\vec{u} \times \vec{v} = (1, 0, 1) \times (1, 1, 0) = (0 - 1, 1 - 0, 1 - 0) = (-1, 1, 1)$$

We'll call the projection  $\text{Proj}_{\vec{v}}(\vec{u})$ . To compute it, we want to compute  $|\vec{u}| \cos \theta$  times a unit vector in the same direction as  $\vec{v}$ , where  $\theta$  is the angle between  $\vec{u}$  and  $\vec{v}$ . We have

$$\text{Proj}_{\vec{v}}(\vec{u}) = |\vec{u}| \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{1}{2} \vec{v} = (1/2, 1/2, 0).$$

(b)  $\vec{u} = (0, 1, 0), \vec{v} = (0, 1, -1)$

$$\vec{u} \cdot \vec{v} = 0 + 1 + 0 = 1$$

$$\vec{u} \times \vec{v} = (0, 1, 0) \times (0, 1, -1) = (-1 - 0, 0 - 0, 0 - 0) = (-1, 0, 0)$$

$$\text{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{1}{2} \vec{v} = (0, 1/2, -1/2)$$

2. Let  $\Pi_1$  be the plane through  $(1, 1, 1)$  with normal vector  $\vec{n}_1 = (1, -1, 0)$ , and let  $\Pi_2$  be the plane defined by  $x + y + z = 1$ .

(a) Write down a linear equation for  $\Pi_1$ .

Denote the normal to  $\Pi_1$  as  $\vec{n}_1 = (1, -1, 0)$ . Then  $(x, y, z)$  is in  $\Pi_1$  if and only if

$$((x, y, z) - (1, 1, 1)) \cdot \vec{n}_1 = 0 \Leftrightarrow x = y.$$

(b) Find the cosine angle between  $\Pi_1$  and  $\Pi_2$ .

The angle  $\theta$  between  $\Pi_1$  and  $\Pi_2$  is the same as the angle between  $\vec{n}_1$  and  $\vec{n}_2$ , where  $\vec{n}_2 = (1, 1, 1)$  is the normal to  $\Pi_2$ . Thus we have

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = 0 \Leftrightarrow \theta = \frac{\pi}{2}.$$

(c) Parameterize the line  $l$  of intersection between  $\Pi_1$  and  $\Pi_2$ .

The direction vector of this line is

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = (1, -1, 0) \times (1, 1, 1) = (-1, -1, 2).$$

Next we find a common point, which amounts to finding a simultaneous solution of the following two equations:

$$x = y, \quad x + y + z = 1.$$

One can check that  $(1/2, 1/2, 0)$  works. Then the line  $l$  is given by

$$l(t) = (1/2, 1/2, 0) + t(-1, -1, 2).$$

(d) Find the distance between  $\Pi_2$  and  $(1, 1, 1)$ .

We first pick a base point  $(1, 0, 0)$  is  $\Pi_2$  (it doesn't matter which basepoint we pick). Then a displacement vector from  $(1, 1, 1)$  to  $\Pi_2$  is  $(1, 1, 1) - (1, 0, 0) = (0, 1, 1)$ , and we can compute the distance as an orthogonal projection of this displacement vector:

$$\text{dist} = |\text{Proj}_{\vec{n}_2}(0, 1, 1)| = \frac{|(0, 1, 1) \cdot \vec{n}_2|}{|\vec{n}_2|} = \frac{2}{\sqrt{3}}.$$

3. Consider the curve  $c(t) = (\cos(2t), \sin(t))$  for  $0 \leq t \leq 2\pi$ .

(a) Sketch this curve.

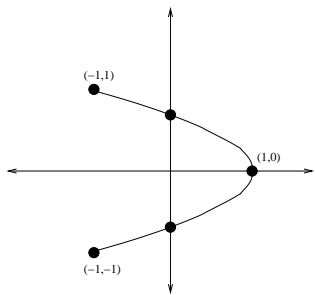


Figure 1: figure for problem 3

The marked points in this sketch correspond are  $(1, 0)$  (corresponding to  $t = 0, \pi$ ),  $(-1, 1)$  (corresponding to  $t = \pi/2$ , where the curve turns around),  $(-1, -1)$  (corresponding to  $t = 3\pi/2$ , where the curve turns around again),  $(0, 1/\sqrt{2})$  (corresponding to  $t = \pi/4, 3\pi/4$ ), and  $(0, -1/\sqrt{2})$  (corresponding to  $t = 5\pi/4, 7\pi/4$ ).

(b) Write down the tangent line to  $c$  at the point  $(-1, 1)$ . (This point corresponds to the parameter value  $t = \pi/2$ .)

We can see from the sketch that  $c$  doesn't really have a tangent line at that point, because it turns around there. We can also see this by looking at  $c'$ . First we compute the velocity vector:  $c'(t) = (-2\sin(2t), \cos(t))$ . Evaluating this at  $t = \pi/2$ , we get a direction vector for the tangent line  $\vec{v} = (-2\sin(\pi), \cos(\pi/2)) = (0, 0)$ . This means there is no tangent line.

(c) Is the tangent line to this curve ever parallel to the line  $y = -x$ ? Be sure to explain your answer.

The tangent line is parallel to  $y = -x$  precisely when  $c'(t)$  is parallel to  $(1, -1)$ , which is a vector parallel to  $y = -x$ . This happens when  $\cos t = 2\sin(2t) = 4\cos t \sin t$ . There are two possibilities: either  $\cos t = 0$ , where  $c$  doesn't really have a tangent line, or  $\sin t = 1/4$ . The latter equation has two solutions,  $t \approx .2527, 2.889$ , so the tangent line to  $c$  is parallel to  $y = -x$  for those values of  $t$ .

(d) Set up, but do not evaluate, the integral to compute the arclength of this curve.

$$\text{Length}(c) = \int_0^{2\pi} |c'(t)| dt = \int_0^{2\pi} \sqrt{4\sin^2(2t) + \cos^2 t} dt$$

4. Consider the function

$$f(x, y) = \int_y^{e^x} \sqrt{1+t^2} dt.$$

(a) Compute the partial derivatives of  $f$ .

We use the fundamental theorem of calculus:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_y^{e^x} \sqrt{1+t^2} dt = \sqrt{1+e^{2x}}(e^x)' = e^x \sqrt{1+e^{2x}}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_y^{e^x} \sqrt{1+t^2} dt = -\sqrt{1+y^2},$$

(b) Does  $f$  have any critical points? Be sure to explain your answer.

Notice that  $\partial f/\partial x > 0$  and  $\partial f/\partial y < 0$ , so there are no points satisfying  $(\partial f/\partial x, \partial f/\partial y) = (0, 0)$ . Thus there are no critical points.

5. Consider the function  $f(x, y) = x^2y - xy^3$ .

(a) Does  $f$  have an upper or lower bound? Explain your answer.

Notice that if we plug in  $y = x$ , we get  $x^3 - x^4 \rightarrow -\infty$  as  $|x| \rightarrow \infty$ . Also, if we plug in  $y = -x$  we get  $x^4 - x^3 \rightarrow \infty$  as  $|x| \rightarrow \infty$ . Thus  $f$  has neither a lower bound nor an upper bound.

(b) Compute the partial derivatives of  $f$ .

$$\frac{\partial f}{\partial x} = 2xy - y^3, \quad \frac{\partial f}{\partial y} = x^2 - 3xy^2$$

(c) Compute the directional derivative of  $f$  in the  $(1/\sqrt{2}, -1/\sqrt{2})$  direction, at the point  $(1, 1)$ . First evaluate the partial derivatives at  $(1, 1)$ :

$$\frac{\partial f}{\partial x}(1, 1) = 2 - 1 = 1, \quad \frac{\partial f}{\partial y}(1, 1) = 1 - 3 = -2.$$

Then the directional derivative is

$$\nabla_{(1/\sqrt{2}, -1/\sqrt{2})} f(1, 1) = (1, -2) \cdot (1/\sqrt{2}, -1/\sqrt{2}) = 3/\sqrt{2}.$$

(d) Find the direction of steepest ascent for  $f$ , starting at  $(1, 1)$ . Make sure to write down a unit vector.

The gradient  $\nabla f = (\partial f/\partial x, \partial f/\partial y)$  always points in the direction of steepest ascent. The unit vector in this direction is

$$\frac{\nabla f(1, 1)}{|\nabla f(1, 1)|} = (1/\sqrt{5}, -2/\sqrt{5}).$$

(e) Notice that  $f(1, 1) = 0$ . Write down the equation of the tangent line to the  $f = 0$  level set, at the point  $(1, 1)$ .

The level sets are always perpendicular to  $\nabla f$ . So we choose a vector perpendicular to  $\nabla f(1, 1) = (1, -2)$ , namely  $\vec{v} = (2, 1)$ . Then the tangent line is parameterized as

$$l(t) = (1, 1) + t(2, 1),$$

or equivalently as

$$y - 1 = \frac{1}{2}(x - 1).$$

6. Consider the function

$$f(x, y) = e^{xy^3 - x^2}.$$

(a) Find and classify all the critical points of  $f$ .

First we take the partial derivatives:

$$\frac{\partial f}{\partial x} = (y^3 - 2x)e^{xy^3 - x^2}, \quad \frac{\partial f}{\partial y} = 3xy^2 e^{xy^3 - x^2}.$$

The critical points occur when  $\nabla f = (\partial f/\partial x, \partial f/\partial y) = (0, 0)$ . Since the exponential is never zero, we require  $y^3 - 2x = 0$  and  $3xy^2 = 0$ . The second equation implies either  $x = 0$  or  $y = 0$ . The first equation implies that either  $x$  or  $y$  is zero, the so is the other one. Thus the only critical point is  $(0, 0)$ .

Next we have to see what kind of critical point it is. We will need the second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2}(0, 0) = (2 + (y^3 - 2x)^2)e^{xy^3 - x^2} \Big|_{(0,0)} = 2, \quad \frac{\partial^2 f}{\partial y^2}(0, 0) = (6xy + 9x^2y^4)e^{xy^3 - x^2} \Big|_{(0,0)} = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = (3y^2 + 3xy^2(y^3 - 2x))e^{xy^3 - x^2} \Big|_{(0,0)} = 0.$$

Thus the discriminant is  $D = (\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2) - (\partial^2 f/\partial x \partial y)^2 = 0$ , so the second derivative test is inconclusive. However, we can still classify this critical point by looking at the  $f = 1$  level set. Taking a logarithm, we see that  $f = 1$  is given by the equation  $xy^3 = x^2$ , so either  $x = 0$  or  $x = y^3$ . These two curves cut the plane into four regions, where the exponent  $xy^3 - x^2$  alternates being positive and negative. Thus  $(0, 0)$  is a saddle-type critical point.

(b) Find the absolute minimum of  $f$  restricted to the square  $0 \leq x \leq 1, 0 \leq y \leq 1$ .

Notice that the exponential function is monotone increasing, so it will suffice to find minima of  $g(x, y) = xy^3 - x^2$ . At the only critical point  $(0, 0)$ , we have  $g = 0$  so  $f = 1$ . Also notice that if we restrict to  $x = 0$ , we get  $g = 0$ , and so  $f = 1$  along this edge as well.

Next we restrict to  $x = 1$ , where  $g = y^3 - 1$ . This has a critical point at  $y = 0$ , with value  $g = -1$  (and so  $f = 1/e$ ). We also have  $g(1, 1) = 0$ , and so  $f(1, 1) = 1$ . If we restrict to  $y = 0$ , then we get  $g = -x^2$ , which has a minimum value of  $g = -1$  (and so  $f = 1/e$ ) at  $(1, 0)$ . Finally, we restrict to  $y = 1$ , where  $g = x - x^2$ . This has a critical point at  $x = 1/2$ , where  $g(1/2, 1) = 1/4$  (and  $f(1/2, 1) = e^{1/4}$ ). Thus our candidates for minimal values of  $f$  are  $f = 1$  (along the  $x = 0$  and edge and at  $(1, 1)$ ),  $f = 1/e$  (at  $(1, 0)$ ) and  $f = e^{1/4}$  (at  $(1/2, 1)$ ). The smallest of these candidates is  $f = 1/e$ , which occurs at  $(1, 0)$ .

7. Recall that two tangent directions to the graph of  $f$  are given by the vectors

$$\left(1, 0, \frac{\partial f}{\partial x}\right), \quad \left(0, 1, \frac{\partial f}{\partial y}\right).$$

(a) Compute a normal vector for the graph.

The normal is given by

$$\left(1, 0, \frac{\partial f}{\partial x}\right) \times \left(0, 1, \frac{\partial f}{\partial y}\right) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right).$$

(b) Can the tangent plane of the graph ever be parallel to the  $x - z$  plane? Explain your answer.

The normal to  $x - z = 0$  is  $(0, 1, 0)$ , so we'd want

$$\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = \lambda(0, 1, 0).$$

This is impossible because any multiple of 0 is still 0.

8. Find the absolute maximum and minimum of the function  $f = xy$  on the ellipse  $4x^2 + y^2 = 4$ .

We use Lagrange multipliers. Letting  $g = 4x^2 + y^2$ , we see that the ellipse is the level set  $g = 4$ . Critical points of  $f$  on the ellipse then satisfy

$$\nabla f = \lambda \nabla g, \quad g = 4,$$

where  $\lambda$  is some nonzero constant. We rewrite this system as

$$(y, x) = \lambda(8x, 2y), \quad 4x^2 + y^2 = 4.$$

The first equations combine to give us  $x = 16\lambda^2 x$ . If  $x \neq 0$ , this yields  $\lambda = \pm 1/4$ , which we can then plug into the last equation (using  $y = \pm 2x$ ):

$$4 = 4x^2 + y^2 = 4x^2 + (\pm 2x)^2 = 8x^2 \Leftrightarrow x = \pm \frac{1}{\sqrt{2}}.$$

This gives four critical points  $\pm(1/\sqrt{2}, 1/\sqrt{2})$  and  $\pm(-1/\sqrt{2}, 1/\sqrt{2})$ . We still have to consider the case of  $x = 0$ , but in this case we also have  $y = 0$ , which doesn't satisfy the constraint. One can check that  $f$  achieves its maximum value of  $1/2$  at  $\pm(1/\sqrt{2}, 1/\sqrt{2})$ , while it achieves its minimum value of  $-1/2$  at  $\pm(-1/\sqrt{2}, 1/\sqrt{2})$ .

9. Evaluate  $\int_D \sqrt{4 - x^2 - y^2} dA$  where  $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y > 0\}$ .

We will use polar coordinates. In these coordinates,  $D$  is the region  $1 \leq r \leq 2$  and  $f = \sqrt{4 - r^2}$ . Then

$$\iint_D f dA = \int_0^{2\pi} \int_1^2 r \sqrt{4 - r^2} dr d\theta = -\frac{1}{2} \int_0^{2\pi} \int_3^0 \sqrt{u} du d\theta = 2\pi\sqrt{3}.$$

10. Set up, but do not evaluate  $\int_D x e^{x^2 y} dA$ , where  $D$  is the region bounded by the curves  $y = x^3$  and  $y = x$ . (Be careful of signs.)

We will use vertical slices (integrating with respect to  $y$  first). First notice that the curves intersect at  $(0, 0)$ ,  $(1, 1)$ , and  $(-1, -1)$ . The region  $D$  splits into two parts: the part with  $-1 < x < 0$ , where  $x < x^3$  and the part with  $0 < x < 1$ , where  $x > x^3$ . Thus we have

$$\iint_D x e^{x^2 y} dA = \int_{-1}^0 \int_x^{x^3} x e^{x^2 y} dy dx + \int_0^1 \int_{x^3}^x x e^{x^2 y} dy dx.$$

11. Evaluate  $\int_D \sqrt{1 - x^2} dA$  where  $D$  is the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ .

The triangle  $D$  is bounded by the axes and the line  $x + y = 1$ . We will evaluate this integral using vertical slices (integrating with respect to  $y$  first). Then

$$\begin{aligned} \iint_D \sqrt{1 - x^2} dA &= \int_0^1 \int_0^{1-x} \sqrt{1 - x^2} dy dx = \int_0^1 (1 - x) \sqrt{1 - x^2} dx = \int_0^1 \sqrt{1 - x^2} dx - \int_0^1 x \sqrt{1 - x^2} dx \\ &= \int_0^1 \sqrt{1 - x^2} dx - \frac{1}{2} \int_0^1 \sqrt{u} du = \int_0^1 \sqrt{1 - x^2} dx - \frac{1}{3} u^{3/2} \Big|_0^1 = \int_0^1 \sqrt{1 - x^2} dx - \frac{1}{3} \\ &= \int_0^{\pi/2} \sqrt{1 - \sin^2 \theta} \cos \theta d\theta - \frac{1}{3} = \int_0^{\pi/2} \cos^2 \theta d\theta - \frac{1}{3} = \frac{1}{2} \int_0^{\pi/2} (1 + 2 \cos(2\theta)) d\theta - \frac{1}{3} \\ &= \frac{\pi}{4} + \frac{1}{2} \sin(2\theta) \Big|_0^{\pi/2} - \frac{1}{3} = \frac{\pi}{4} - \frac{1}{3} \end{aligned}$$

12. Consider the domain  $D := \{(x, y) \mid |x + y| \leq 1\}$ .

(a) Sketch  $D$ .

The region  $D$  is the strip between the two lines drawn below.



Figure 2: figure for problem 12

(b) If we change variables by  $u = \frac{1}{\sqrt{2}}(x + y)$ ,  $v = \frac{1}{\sqrt{2}}(-x + y)$ , what is  $D$  in the  $u, v$  coordinate system?

Notice that  $x + y = \sqrt{2}u$ , so  $D$  is the domain  $|u| \leq 1/\sqrt{2}$ .

(c) Set up, but do not evaluate, the integral  $\iint_D \cos(\pi x - \pi y) dA$  in the  $(u, v)$  coordinate system.

Observe that

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{2}} = \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{1}{\sqrt{2}}.$$

So the Jacobian for the change of variables is

$$J = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = \frac{1}{2} + \frac{1}{2} = 1.$$

Also, in these variables,  $\cos(\pi x - \pi y) = \cos(-(\pi\sqrt{2})v) = \cos((\pi\sqrt{2})v)$ . Thus the integral is

$$\iint_D \cos(\pi x - \pi y) dA = \int_{-\infty}^{\infty} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \cos((\pi\sqrt{2})v) du dv.$$

13. Consider the vector field  $\vec{F} = (-y, x)$ .

(a) Sketch  $\vec{F}$ .

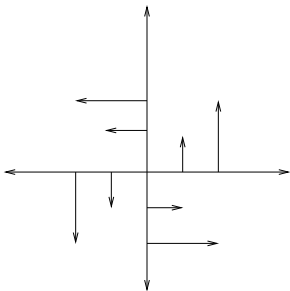


Figure 3: figure for problem 13

(b) Compute  $\int_{\gamma} \vec{F} \cdot d\vec{s}$  where  $\gamma(t) = (\cos t, \sin t)$  for  $0 \leq t \leq 2\pi$ .

Observe that  $\gamma' = (-\sin t, \cos t)$ . So we have

$$\int_{\gamma} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\gamma(t)) \cdot \gamma'(t) dt = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt = \int_0^{2\pi} dt = 2\pi.$$

14. Consider the vector field  $\vec{F} = (2xe^{x^2+y^2}, 2ye^{x^2+y^2} + \cos y)$ .

(a) Show that  $\vec{F} = \nabla f$  for some  $f$ .

We have to check that  $\partial F_2/\partial x = \partial F_1/\partial y$ . We can compute:

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(2ye^{x^2+y^2} + \cos y) = 4xye^{x^2+y^2}, \quad \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(2xe^{x^2+y^2}) = 4xye^{x^2+y^2}.$$

(b) Find a function  $f$  such that  $\vec{F} = \nabla f$ .

We have to find an  $f$  such that

$$\frac{\partial f}{\partial x} = 2xe^{x^2+y^2}, \quad \frac{\partial f}{\partial y} = 2ye^{x^2+y^2} + \cos y.$$

Integrating the first equation with respect to  $x$ , we get  $f = e^{x^2+y^2} + g(y)$  for some unknown function  $g(y)$ . The second equation then says  $g'(y) = \cos y$ , which we can integrate to get  $g = \sin y$ . Thus we have  $f(x, y) = e^{x^2+y^2} + \sin y$  (up to adding a constant).

(c) Compute  $\int_{\gamma} \vec{F} \cdot d\vec{s}$ , where  $\gamma(t) = (\cos t, \sin t)$  for  $0 \leq t \leq \pi/2$ . (Hint: you don't need to actually do an integral.)

The endpoints of  $\gamma$  are  $\gamma(\pi/2) = (0, 1)$  and  $\gamma(0) = (1, 0)$ . Because  $\vec{F} = \nabla f = \nabla(e^{x^2+y^2} + \sin y)$ , we can use the fundamental theorem of calculus:

$$\int_{\gamma} \vec{F} \cdot d\vec{s} = f(0, 1) - f(1, 0) = (e^1 + \sin(1)) - (e^1 + \sin(0)) = \sin(1).$$

15. Consider  $\vec{F} = (-y + x^2 - y \cos(xy), x - y^3 - x \cos(xy))$ .

(a) Is  $\vec{F} = \nabla f$  for some  $f$ ? Be sure to explain your answer.

No. Indeed,  $\partial F_2/\partial x = 1$ , while  $\partial F_1/\partial y = -1$ . These two would have to be equal if  $\vec{F} = \nabla f$ .

(b) Compute  $\int_{\gamma} \vec{F} \cdot d\vec{s}$ . (Hint: don't actually compute the line integral.)

Forgot to say that  $\gamma$  is supposed to be the unit circle (centered at the origin, but that part isn't important). Then  $\gamma = \partial D$ , where  $D$  is the unit disc, so we can use Green's theorem:

$$\int_{\gamma} \vec{F} \cdot d\vec{s} = \int \int_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = 2 \int \int_D dA = 2\pi.$$

16. Consider the vector field  $\vec{F} = (z + x^3 - yze^{xyz}, y - xze^{xyz}, -x + z^2 - xye^{xyz})$ .

(a) Compute  $\nabla \cdot \vec{F}$  and  $\nabla \times \vec{F}$ .

First we compute the curl:

$$\nabla \times \vec{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = (0, 2, 0).$$

Next we compute the divergence:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + 2z + 1 - (x^2y^2 + y^2z^2 + x^2z^2)e^{xyz}.$$

(b) Is  $\vec{F} = \nabla f$  for some  $f$ ? Be sure to explain your answer.

No. If  $\vec{F} = \nabla f$ , then  $\nabla \times \vec{F} = 0$ . However, we just computed that the curl of  $\vec{F}$  is not 0.

(c) Compute  $\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA$ , where  $\Sigma$  is the upper unit hemisphere, centered at  $(0, 0, 0)$ , with the outward unit normal.

Let  $D$  be the unit disc in the  $x - y$  plane (with the upward normal  $\vec{n} = (0, 0, 1)$ ), and observe that  $\Sigma$  and  $D$  have the same boundary curve  $\gamma$ , which is the unit circle in the  $x - y$  plane. Then by Stokes theorem

$$\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA = \int_{\gamma} \vec{F} \cdot d\vec{s} = \int \int_D \nabla \times \vec{F} \cdot \vec{n} dA = \int \int_D (0, 2, 0) \cdot (0, 0, 1) dA = 0.$$

17. Consider the surface  $\vec{r}(u, v) = (u \cos(v), u \sin(v), v)$ .

(a) Compute the tangent vectors  $\partial\vec{r}/\partial u$  and  $\partial\vec{r}/\partial v$ .

$$\frac{\partial\vec{r}}{\partial u} = (\cos v, \sin v, 0), \quad \frac{\partial\vec{r}}{\partial v} = (-u \sin v, u \cos v, 1).$$

(b) Verify that this is a good parameterization, by checking that  $\partial\vec{r}/\partial u$  and  $\partial\vec{r}/\partial v$  are never parallel.

We compute the cross product  $(\partial\vec{r}/\partial u) \times (\partial\vec{r}/\partial v)$ , and check that we don't get zero:

$$\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v} = (\cos v, \sin v, 0) \times (-u \sin v, u \cos v, 1) = (\sin v, -\cos v, u).$$

Notice that  $\sin v$  and  $\cos v$  are never simultaneously zero, so either the first or the second component of this vector is nonzero.

(c) Find the equation of the tangent plane to  $\vec{r}$  for the parameter values  $u = 1, v = \pi$ .

First evaluate the normal vector we computed above for the parameter values  $u = 1, v = \pi$ . We get

$$\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v} = (0, 1, 1).$$

Next we evaluate  $\vec{r}(1, \pi) = (-1, 0, \pi)$ . The equation of the tangent plane is then

$$0 = (x - (-1), y - 0, z - \pi) \cdot (0, 1, 1) \Leftrightarrow y + z = \pi.$$

(d) Is the tangent plane ever parallel to the  $x - y$  plane? Be sure to explain your answer.

The tangent plane to the surface is parallel to the  $x - y$  plane precisely when the normal is parallel to  $(0, 0, 1)$ . Thus we would need

$$(\sin v, -\cos v, u) = \lambda(0, 0, 1)$$

for some  $\lambda \neq 0$ . However, there is no value of  $v$  such that  $\sin v = 0$  and  $\cos v = 0$ , so this never happens, and the tangent plane is never parallel to the  $x - y$  plane.

(e) What is this surface? Can you draw a sketch of it? (Hint: fix a value of  $u$ , for instance  $u = 1$  or  $u = 0$ , and draw the resulting curve.)

This is a helicoid. If you fix a value of  $u$ , the curve you get is a helix, twisting around the  $z$  axis. (Except  $u = 0$ ; then you just get the  $z$  axis.) The easiest way to visualize this surface is to start with the line  $y = z = 0$ , and raise it at a constant rate, while you twist it in the counter-clockwise direction around the  $z$  axis at the same rate. This sweeps out the surface.

18. Compute  $\int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA$ , where  $\vec{F} = (-y, x, 0)$  and  $\Sigma$  is the upper unit hemisphere, centered at the origin, with the outward unit normal.

There are a couple of ways to do this. We'll use Stokes theorem and parameterize  $\partial\Sigma$  as  $\gamma(t) = (\cos t, \sin t, 0)$  for  $0 \leq t \leq 2\pi$ . Observe that  $\gamma'(t) = (-\sin t, \cos t, 0)$ . Then

$$\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA = \int_{\gamma} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0) dt = \int_0^{2\pi} dt = 2\pi.$$

19. Compute  $\int_{\Sigma} \vec{F} \cdot \vec{n} dA$ , where  $\vec{F} = (x + yz - \cos y, y - e^{xz} + z^2, z - x \cos(x^2 y))$  and  $\Sigma$  is the unit sphere (centered at the origin) with the outward unit normal.

Here we use the divergence theorem. Notice that  $\Sigma = \partial B$ , where  $B$  is the unit ball centered at the origin. Also,  $\nabla \cdot \vec{F} = 3$ . Then

$$\int \int_{\Sigma} \vec{F} \cdot \vec{n} dA = \int \int_{\partial B} \vec{F} \cdot \vec{n} dA = \int \int \int_B \nabla \cdot \vec{F} dV = 3 \int \int \int_B dV = 4\pi.$$

20. Compute  $\int_{\Sigma} \vec{F} \cdot \vec{n} dA$ , where  $\vec{F} = (x + ze^{y^2+z}, y - z \cos(x+z^2), z)$  and  $\Sigma$  is the upper unit hemisphere, centered at the origin, with the outward unit normal. (Hint: what is  $\vec{F}$  restricted to the plane  $z = 0$ ?)

Notice that  $\nabla \cdot \vec{F} = 3$ . We'd like to use the divergence theorem, but  $\Sigma$  is not a closed surface, so it's not the boundary of any solid region. However, we can add something to  $\Sigma$  and make the resulting union of surfaces into the boundary of a solid. In this case, we add the unit disc  $D = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ , with the

**downward** normal. Then we let  $B_+ = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, z \geq 0\}$  be the upper half-ball. Observe that  $\partial B_+ = \Sigma + D$ . Then by the divergence theorem, we have

$$\int \int_{\Sigma} \vec{F} \cdot \vec{n} dA + \int \int_D \vec{F} \cdot \vec{n} dA = \int \int \int_{B_+} \nabla \cdot \vec{F} dV = \int \int \int_{B_+} 3dV = 2\pi.$$

All that remains is to compute  $\int \int_D \vec{F} \cdot \vec{n} dA$ . We have

$$\int \int_D \vec{F} \cdot \vec{n} dA = \int_0^{2\pi} \int_0^1 (r \cos \theta, r \sin \theta, 0) \cdot (0, 0, -1) r dr d\theta = 0.$$

Putting everything together, we see that

$$\int \int_{\Sigma} \vec{F} \cdot \vec{n} dA = \int \int \int_{B_+} \nabla \cdot \vec{F} dV - \int \int_D \vec{F} \cdot \vec{n} dA = 2\pi.$$

21. Let  $\Pi_1$  be the plane through  $(1, 0, 1)$  with normal  $\vec{n}_1 = (1, -1, 0)$ , and let  $\Pi_2$  be the plane given by  $x = y + z$ .

(a) (3 points) Find  $\cos \theta$ , where  $\theta$  is the angle between  $\Pi_1$  and  $\Pi_2$ .

The angle between the planes is the same as the angle between the two normal vectors  $\vec{n}_1 = (1, -1, 0)$  and  $\vec{n}_2 = (1, -1, -1)$ . The cosine of this angle is given by

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|} = \frac{(1, -1, 0) \cdot (1, -1, -1)}{\|(1, -1, 0)\| \|(1, -1, -1)\|} = \frac{2}{\sqrt{6}}.$$

(b) (4 points) Find a linear equation for the plane  $\Pi_1$ .

This linear equation is

$$\vec{n}_1 \cdot (x, y, z) = \vec{n}_1 \cdot (x_0, y_0, z_0) \Leftrightarrow (1, -1, 0) \cdot (x, y, z) = (1, -1, 0) \cdot (1, 0, 1) \Leftrightarrow x - y = 1.$$

(c) (4 points) Parameterize the line  $l$  of intersection between  $\Pi_1$  and  $\Pi_2$ .

We need to find two things to determine this line: a direction vector (a vector parallel to both planes) and point on both planes (a common solution to the two equations). Let's find the direction vector first. This vector  $\vec{v}$  has to be perpendicular to both normals, so we choose

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = (1, -1, 0) \times (1, -1, -1) = (1, 1, 0),$$

Next we find a common solution to the two equations:

$$1 = x - y = z.$$

Notice the line of intersection is horizontal (it lies in the  $z = 1$  plane), so we can choose either  $x$  or  $y$ . We choose  $y = 0$ , which forces  $x = 1$ . Thus a point on the line is  $(1, 0, 1)$  and a parameterization of the line is

$$l(t) = (1, 0, 1) + t(1, 1, 0).$$

(d) (4 points) Find the distance between  $(1, 1, 2)$  and  $\Pi_2$ .

First we need to find a base point on the plane  $\Pi_2$ . The point  $(1, 0, 1)$  we found in the previous part works just fine. Next we find the scalar projection of  $\vec{PQ} = (1, 1, 2) - (1, 0, 1) = (0, 1, 1)$  onto the normal vector  $\vec{n}_2 = (1, -1, -1)$ . This is the distance we're looking for:

$$d = \frac{|\vec{PQ} \cdot \vec{n}_2|}{\|\vec{n}_2\|} = \frac{|(0, 1, 1) \cdot (1, -1, -1)|}{\|(1, -1, -1)\|} = \frac{2}{\sqrt{3}}.$$

22. Consider  $f(x, y) = x^2y - xy - x^2 - 6y - x + 6$ .

(a) (5 points) Verify that the only critical points of  $f$  are  $(3, 7/5)$  and  $(-2, 3/5)$ .

We want to find the points  $(x, y)$  where  $\nabla f = (\partial f / \partial x, \partial f / \partial y) = (0, 0)$ . These equations are

$$0 = \frac{\partial f}{\partial x} = 2xy - y - 2x - 1, \quad 0 = \frac{\partial f}{\partial y} = x^2 - x - 6.$$

We can factor the second equation:  $0 = (x - 3)(x + 2)$ , so either  $x = 3$  or  $x = -2$ . If  $x = 3$  then the first equation reads  $0 = 5y - 7$ , so  $y = 7/5$ . If  $x = -2$  then the first equation reads  $0 = -5y + 3$ , so  $y = 3/5$ . Thus the only two critical points are  $(3, 7/5)$  and  $(-2, 3/5)$ .

- (b) (5 points) Classify these critical points as local maxima, local minima, saddle points, or none of the above. We will need to evaluate the second partial derivatives of  $f$  to apply the second derivative test. We have

$$\frac{\partial^2 f}{\partial x^2} = 2y - 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 2x - 1, \quad \frac{\partial^2 f}{\partial y^2} = 0.$$

Thus the discriminant is

$$D = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 = -(2x - 1)^2.$$

Evaluating  $D$  at the critical points, we see

$$D(3, 7/5) = -25 < 0, \quad D(-2, 3/5) = -9 < 0,$$

so both critical points are saddle points.

- (c) (5 points) Observe that  $f(1, 1) = -2$ . Write down an equation for the tangent line to the level set  $\{f = -2\}$ , at the point  $(1, 1)$ .

Recall that the tangent lines to level curves are perpendicular to  $\nabla f$ . Here,  $\nabla f(1, 1) = (-2, -6)$ , so the tangent line has slope  $-2/6 = -1/3$ . Thus the tangent line has the equation

$$y - 1 = -\frac{1}{3}(x - 1) \Leftrightarrow y = -\frac{x}{3} + \frac{4}{3}.$$

23. Recall that two tangent directions to the graph of a function  $f(x, y)$  are

$$\left(1, 0, \frac{\partial f}{\partial x}\right), \quad \left(0, 1, \frac{\partial f}{\partial y}\right).$$

- (a) (5 points) Write down a normal vector to the tangent plane of the graph of  $f$ .

The normal vector will be the cross product of the two tangent vectors:

$$\vec{n} = \left(1, 0, \frac{\partial f}{\partial x}\right) \times \left(0, 1, \frac{\partial f}{\partial y}\right) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right).$$

- (b) (5 points) Is it ever possible that the tangent plane to the graph of  $f$  is parallel to the plane  $x + y = 0$ ? Be sure to explain your answer.

No. The plane  $x + y = 0$  has a normal  $\vec{N} = (1, 1, 0)$ , which cannot be a scalar multiple of  $\vec{n} = (-\partial f/\partial x, -\partial f/\partial y, 1)$ , because the third component of  $\vec{N}$  is zero, while the third component of  $\vec{n}$  is nonzero.

24. Consider the composition  $g(t) = f(x(t), y(t))$ , where  $x(1) = 1$ ,  $x'(1) = -1$ ,  $y(1) = 2$ ,  $y'(1) = 3$ , and  $\nabla f(1, 2) = (3, -2)$ .

- (a) (3 points) What is  $g'(1)$ ?

By the chain rule,

$$g'(1) = \nabla f(x(1), y(1)) \cdot (x'(1), y'(1)) = \nabla f(1, 2) \cdot (-1, 3) = (3, -2) \cdot (-1, 3) = -9.$$

- (b) (4 points) Suppose  $x'(2) = 0$  and  $y'(2) = 0$ . Is it necessarily true that  $g'(2) = 0$ ? Explain your answer.

Yes:

$$g'(2) = \nabla f(x(2), y(2)) \cdot (x'(2), y'(2)) = \nabla f \cdot (0, 0) = 0.$$

- (c) (3 points) Suppose  $g'(-1) = 0$ . Is it necessarily true that  $x'(-1) = 0$  and  $y'(-1) = 0$ ? Explain your answer.

No. You could have  $\nabla f(x(-1), y(-1)) = (0, 0)$ . You could also have some cancelation, such as  $x'(-1) = 1$ ,  $y'(-1) = 1$ ,  $\nabla f = (1, 1)$ .

25. (a) (5 points) Evaluate  $\iint_D e^{x^2+y^2} dA$ , where  $D$  is the domain  $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, x \leq 0\}$ .

We use polar coordinates:  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Then the domain is  $1 \leq r \leq 2$  and the function is  $f = e^{r^2}$ . Also recall that the area element is  $dA = r dr d\theta$ . Then

$$\iint_D e^{x^2+y^2} dA = \int_0^{2\pi} \int_1^2 r e^{r^2} dr d\theta = 2\pi \int_1^2 r e^{r^2} dr = \pi e^{r^2} \Big|_1^2 = \pi(e^4 - e).$$

- (b) (5 points) Set up, but **do not evaluate** the integral  $\int \int_D e^{x^2 y} \sin(y^3 + x) dA$ , where  $D$  is the region bounded by the line  $y = 1 + x$  and the curve  $y = x^2$ . (It might help to draw a picture.)

The curve  $y = x^2$  is a parabola opening to the up, which intersects the line  $y = 1 + x$  at  $x^2 = x + 1$ , or  $0 = x^2 - x - 1$ . Using the quadratic formula, we get  $x = (1 \pm \sqrt{5})/2$ , and so  $y = (3 \pm \sqrt{5})/2$ . Also, for  $(1 - \sqrt{5})/2 \leq x \leq (1 + \sqrt{5})/2$  the line lies above the parabola. We will set up the integral using vertical slices (b/c using horizontal slices we'd have to set up two integrals):

$$\int \int_D e^{x^2 y} \sin(y^3 + x) dA = \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} \int_{x^2}^{1+x} e^{x^2 y} \sin(y^3 + x) dy dx.$$

26. Consider the vector field  $\vec{F} = (y \cos(xy), x \cos(xy) + y^2)$  in the plane.

- (a) (5 points) Verify that  $\vec{F}$  is the gradient of a function.

We test that  $\partial F_2 / \partial x = \partial F_1 / \partial y$ :

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x} (x \cos(xy) + y^2) = \cos(xy) - xy \sin(xy), \quad \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y} (y \cos(xy)) = \cos(xy) - xy \sin(xy).$$

- (b) (5 points) Find a function  $f$  such that  $\vec{F} = \nabla f$ .

We want  $\partial f / \partial x = y \cos(xy)$ , so we try  $f(x, y) = \sin(xy) + g(y)$  (integrating in  $x$ ). Now we compare to  $\partial f / \partial y$ :

$$x \cos(xy) + g' = F_2 = x \cos(xy) + y^2 \Leftrightarrow g' = y^2.$$

Thus we can choose  $g(y) = y^3/3$ , and so

$$f = \sin(xy) + \frac{y^3}{3}.$$

- (c) (5 points) Evaluate  $\int_{\gamma} \vec{F} \cdot d\vec{s}$  where  $\gamma(t) = (t, 1 - t^2)$  for  $0 \leq t \leq 1$ . (Hint: you do not need to compute an integral.)

Note that  $\gamma(0) = (0, 1)$  while  $\gamma(1) = (1, 0)$ . Then by the Fundamental Theorem of Calculus, we have

$$\int_{\gamma} \vec{F} \cdot d\vec{s} = f(1, 0) - f(0, 1) = 0 - \frac{1}{3} = -\frac{1}{3}.$$

27. Consider the vector field  $\vec{F} = (y + yze^{xyz}, -x + xze^{xyz}, xye^{xyz})$  in three-space.

- (a) (5 points) Compute the curl of  $\vec{F}$ .

We plug in the formula for the curl of a vector field and see

$$\nabla \times \vec{F} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = (0, 0, -2).$$

- (b) (5 points) Is  $\vec{F}$  the gradient of a function? Be sure to explain your answer.

No. The curl of a gradient is always zero.

- (c) (5 points) Evaluate  $\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA$ , where  $\Sigma$  is the hemisphere  $\{(x, y, z) \mid x^2 + y^2 + z^2, x \geq 0\}$ , oriented with the normal  $\vec{n}$  pointing away from the origin. (Hint: you have to compute an integral, but not necessarily over  $\Sigma$ .)

Notice that  $\Sigma$  has the same boundary as the unit disc  $D$  in the  $xy$ -plane, centered at the origin, oriented with the upward unit normal  $\vec{N} = (0, 0, 1)$ . Then we can use Stokes theorem as follows:

$$\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA = \int_{\partial \Sigma} \vec{F} \cdot d\vec{s} = \int_{\partial D} \vec{F} \cdot d\vec{s} = \int \int_D \nabla \times \vec{F} \cdot \vec{N} dA = \int \int_D (0, 0, -2) \cdot (0, 0, 1) dA = -2 \int \int_D dA = -2\pi.$$

28. Consider the vector field  $\vec{F} = (\cos(y^2 z^{1/3}) - x + \ln y, y - e^{xz^3} + z, z + \sin(x^2 y^6) \ln(\sqrt{1 + x^2 + y^2}))$  in three-space.

(a) (5 points) Compute the divergence of  $\vec{F}$ .

We use the formula for divergence:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1.$$

(b) (5 points) Evaluate  $\int \int_{\Sigma} \vec{F} \cdot \vec{n} dA$ , where  $\Sigma$  is the unit sphere, centered at the origin, oriented with the outward normal  $\vec{n}$ . (Hint: think before you compute.)

We use the divergence theorem, and the fact that the unit sphere  $\Sigma$  bounds the unit ball  $B$ , which has volume  $4\pi/3$ :

$$\int \int_{\Sigma} \vec{F} \cdot \vec{n} dA = \int \int \int_B \nabla \cdot \vec{F} dV = \int \int \int_B dV = \frac{4\pi}{3}.$$

29. Consider the curve  $c(t) = (\cos t, \sin t, t)$ .

(a) (4 points) Find the equation of the tangent line to  $c$  at time  $t = \pi$ .

In general the tangent vector to  $c$  at time  $t_0$  is

$$c'(t_0) = (-\sin(t_0), \cos(t_0), 1).$$

Evaluating at  $t_0 = \pi$ , we have a tangent vector  $c'(\pi) = (0, -1, 1)$ . Also observe that  $c(\pi) = (-1, 0, \pi)$ . Putting this all together, we see that the equation for the tangent line is

$$l(t) = c(\pi) + t c'(\pi) = (-1, 0, \pi) + t(0, -1, 1) = (-1, -t, \pi + t).$$

(b) (3 points) Is the tangent line to  $c$  ever parallel to the  $xy$ -plane? Be sure to explain your answer.

The normal to the  $xy$ -plane is  $\vec{n} = (0, 0, 1)$ . In order for the tangent line to  $c$  to be parallel to the  $xy$ -plane, the line would have to be perpendicular to  $(0, 0, 1)$ , which is equivalent to  $c'(t) \cdot (0, 0, 1) = 1$ . However,

$$c'(t) \cdot (0, 0, 1) = (-\sin t, \cos t, 1) \cdot (0, 0, 1) = 1 \neq 0,$$

so the answer is no.

(c) (3 points) Find the length of the section of  $c$  corresponding to  $0 \leq t \leq 2\pi$ .

Notice that

$$|c'(t)| = |(-\sin t, \cos t, 1)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Thus, the length is

$$\text{length} = \int_0^{2\pi} |c'(t)| dt = \int_0^{2\pi} \sqrt{2} dt = \sqrt{2} \cdot 2\pi.$$

30. Consider the function  $f(x, y) = x^3 - x^2 y + 3xy$ .

(a) (5 points) Verify that the only critical points of this function are  $(0, 0)$ ,  $(3, 9)$ .

A critical point  $(x_0, y_0)$  will satisfy  $\nabla f(x_0, y_0) = (0, 0)$ . Computing the gradient, we see

$$\nabla f = (3x^2 - 2xy + 3y, -x^2 + 3x) = (0, 0).$$

Setting the second component to zero, we get  $0 = -x^2 + 3x$ , which happens if and only if  $x = 0$  or  $x = 3$ . We plug both these possibilities into the first component. If  $x = 0$  then we get  $0 = 3y \Leftrightarrow y = 0$ , and so  $(0, 0)$  is a critical point. If  $x = 3$ , we get  $0 = 27 - 6y + 3y$ , so  $y = 9$  and the other critical point is  $(3, 9)$ .

(b) (5 points) Classify these critical points as local minima, local maxima, saddle points, or none of the above.

We will need to evaluate some second derivatives to apply the second derivative test:

$$\frac{\partial^2 f}{\partial x^2} = 6x - 2y, \quad \frac{\partial^2 f}{\partial x} \partial y = -2x + 3, \quad \frac{\partial^2 f}{\partial y^2} = 0.$$

Then the discriminant is

$$D = \left( \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} \right) - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 = -(3 - 2x)^2.$$

Evaluate this at each critical point:

$$D(0, 0) = -9 < 0, \quad D(3, 9) = -9 < 0,$$

so both the critical points are saddle points.

- (c) (5 points) Observe that  $f(1, 1) = 3$ . Write down an equation for the tangent line to the  $f = 3$  level set of at  $(1, 1)$ .

Recall that the gradient is always perpendicular to level sets. Evaluating at  $(1, 1)$ , we have  $\nabla f(1, 1) = (3 - 2 + 3, -1 + 3) = (4, 2)$ , so the slope of the perpendicular line is  $2/4 = 1/2$ . This means the slope of the tangent line is  $-2$ , and so the equation of the tangent line is

$$y - 1 = -2(x - 1) \Leftrightarrow y = -2x + 3.$$

31. Consider the composition  $g(t) = f(x(t), y(t))$ , where  $x(1) = 1$ ,  $y(1) = 2$ ,  $x'(1) = -1$ ,  $y'(1) = 1$  and  $\nabla f(1, 2) = (3, 2)$ .

- (a) (4 points) What is  $g'(1)$ ?

Use the chain rule:

$$g'(1) = \nabla f|_{(1,2)} \cdot (x'(1), y'(1)) = (3, 2) \cdot (-1, 1) = -1.$$

- (b) (3 points) Suppose  $x'(2) = 0$  and  $y'(2) = 0$ . Is it necessarily true that  $g'(2) = 0$ ? Be sure to explain your answer.

Yes. By the chain rule,

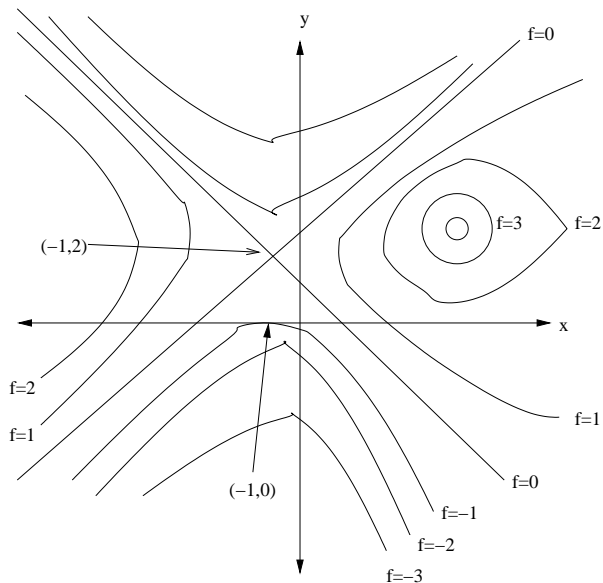
$$g'(2) = \frac{\partial f}{\partial x} x'(2) + \frac{\partial f}{\partial y} y'(2) = \frac{\partial f}{\partial x} \cdot 0 + \frac{\partial f}{\partial y} \cdot 0 = 0,$$

regardless of what  $\partial f/\partial x$  and  $\partial f/\partial y$  are.

- (c) (3 points) Suppose  $g'(0) = 0$ . Is it necessarily true that  $x'(0) = 0$  and  $y'(0) = 0$ ? Be sure to explain your answer.

No. You could, for instance, have  $x'(0) = -1$ ,  $y'(0) = 1$ , and  $\partial f/\partial x = 1 = \partial f/\partial y$ . Then, even though neither  $x'(0) = 0$  nor  $y'(0) = 0$ , we still have  $g'(0) = -1 + 1 = 0$ .

32. Below is a sketch of some of the level curves of a function  $f = f(x, y)$ .



- (a) (5 points) In which direction does  $\nabla f(-1, 0)$  point? Be sure to explain your answer.

First recall that  $\nabla f$  is perpendicular to the level sets, and observe that the level curve has a horizontal tangent line through  $(-1, 0)$ . This means  $\nabla f$  must either point up or down. Also, the function values are increasing as we move up, for  $\nabla f(-1, 0)$  must point up (in the  $(0, 1)$  direction).

- (b) (5 points) If  $(-1, 2)$  is a critical point of  $f$ , what kind of critical point do you expect it to be? Be sure to explain your answer.

One would expect  $(-1, 2)$  to be a saddle point. First of all, two branches of the  $f = 0$  level set cross at  $(-1, 2)$ . Also, the function values increase if you move to the right or left, but decrease if you move down.

33. (5 points) Set up, but **do not evaluate**, the integral  $\int \int_D x \ln(1 + x^2 + y^2) dA$  as an iterated integral, where  $D$  is the domain bounded by  $x = y^2 - 2y$  and  $y = x - 1$ . (It might help to draw a picture.)

If you sketch the boundary curves of the domain of integration, we get a parabola opening to the right and a line of slope 1. We'll first find the intersection points by finding simultaneous solutions to the equations:

$$x = y^2 - y = y + 1 \Leftrightarrow 0 = y^2 - 2y - 1 \Leftrightarrow y = 1 \pm \sqrt{2}.$$

If  $y = 1 + \sqrt{2}$  then  $x = 2 + \sqrt{2}$ , while if  $y = 1 - \sqrt{2}$  then  $x = 2 - \sqrt{2}$ . So the intersection points are  $(2 \pm \sqrt{2}, 1 \pm \sqrt{2})$ .

We'll set the integral up by taking horizontal slices, integrating with respect to  $x$  first (if we integrate with respect to  $y$  first we will have to write several iterated integrals). Notice that the parabola lies to the left of the line, in the region we're interested in. The integral is

$$\iint_D x \ln(1 + x^2 + y^2) dA = \int_{1-\sqrt{2}}^{1+\sqrt{2}} \int_{y^2-y}^{y+1} x \ln(1 + x^2 + y^2) dx dy.$$

34. (10 points) Evaluate  $\int \int_D \sqrt{x^2 + y^2 - 1} dA$ , where  $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4\}$ .

Here we change variables, using polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Then  $x^2 + y^2 = r^2$ , and the domain is  $D = \{1 \leq r \leq 2\}$ . Also, the area element is  $dA = r dr d\theta$ . We compute:

$$\iint_D \sqrt{x^2 + y^2 - 1} dA = \int_0^{2\pi} \int_1^2 \sqrt{r^2 - 1} r dr d\theta = 2\pi \int_1^2 r \sqrt{r^2 - 1} dr = \pi \int_0^3 \sqrt{u} du = \frac{2\pi}{3} \cdot 3^{3/2} = 2\pi\sqrt{3}.$$

35. Consider the vector field  $\vec{F} = (ye^{xy} + 2x, xe^{xy})$  in the plane.

- (a) (5 points) Verify that  $\vec{F} = \nabla f$  for some function  $f = f(x, y)$ .

Because  $F$  is defined everywhere, without any singularities, it suffices to check  $\partial F_1 / \partial y = \partial F_2 / \partial x$ :

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(ye^{xy} + 2x) = e^{xy} + xye^{xy}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(xe^{xy}) = e^{xy} + xye^{xy}.$$

- (b) (5 points) Find a function  $f$  such that  $\vec{F} = \nabla f$ .

We want  $\partial f / \partial y = xe^{xy}$ , so we can choose  $f(x, y) = e^{xy} + g(x)$ . Then matching  $F_1 = \partial f / \partial x = ye^{xy} + g'$ , we see  $g' = 2x$ , so we can choose  $g = x^2$  and

$$f(x, y) = e^{xy} + x^2.$$

- (c) (5 points) Evaluate  $\int_\gamma \vec{F} \cdot d\vec{s}$ , where  $\gamma(t) = (\cos t, \sin t)$  for  $0 \leq t \leq \pi/2$ . (Think before you compute.)

Here we use the fundamental theorem of calculus:

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int_\gamma \nabla f \cdot d\vec{s} = f(\gamma(\pi/2)) - f(\gamma(0)) = f(0, 1) - f(1, 0) = (e^0 + 0^2) - (e^0 + 1^2) = -1.$$

36. Consider the vector field  $\vec{F} = (z \ln(1 + y^2) - x, 2y - xe^{xz}, -1 + xy \sin(1 + x^2 + y^2))$  in three-space.

- (a) (5 points) Compute the divergence of  $\vec{F}$ .

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = -1 + 2 + 0 = 1.$$

- (b) (5 points) Evaluate  $\int \int_\Sigma \vec{F} \cdot \vec{n} dA$ , where  $\Sigma$  is the unit sphere centered at the origin, oriented with the **inward** normal. (Think before you compute.)

Notice that  $\Sigma$ , with the **opposite** orientation, is the boundary of the unit ball  $B = \{x^2 + y^2 + z^2 \leq 1\}$ . By the divergence theorem,

$$\int \int_\Sigma \vec{F} \cdot \vec{n} dA = - \int \int \int_B \nabla \cdot \vec{F} dV = - \int \int \int_B dV = -\text{Vol}(B) = -\frac{4\pi}{3}.$$

37. Consider the vector field  $\vec{F} = (yz \cos(xyz), xz \cos(xyz), y + xy \cos(xyz))$  in three-space.

(a) (5 points) Compute the curl of  $\vec{F}$ .

Notice that  $\vec{F} = \nabla(\sin(xyz)) + (0, 0, y)$ , so

$$\nabla \times \vec{F} = \nabla \times (\nabla(\sin(xyz)) + (0, 0, y)) = \nabla \times (0, 0, y) = (1, 0, 0).$$

(b) (5 points) Is  $\vec{F}$  the gradient of a function  $f(x, y, z)$ ? Be sure to explain your answer.

No. The curl of a gradient is always zero, and  $\nabla \times \vec{F}$  is nonzero.

(c) (5 points) Evaluate  $\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA$ , where  $\Sigma$  is the upper unit hemisphere centered at the origin parameterized by  $\{(\sqrt{1-u^2} \cos v, \sqrt{1-u^2} \sin v, u) \mid 0 \leq u \leq 1, 0 \leq v \leq 2\pi\}$ , oriented with the **upward** normal. (Hint: you don't have to integrate over  $\Sigma$ , you can replace it with another surface which has the same boundary.)

Let  $D$  be the unit disc in the  $xy$ -plane, oriented with the **upward** unit normal  $\vec{N} = (0, 0, 1)$ . Notice that  $\Sigma$  and  $D$  have the same boundary  $\gamma$ , which is the unit circle in the  $xy$ -plane, oriented in the counter-clockwise direction. Then by Stokes theorem

$$\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA = \int_{\gamma} \vec{F} \cdot d\vec{s} = \int \int_D \nabla \times \vec{F} \cdot \vec{N} dA = 0.$$

In the last step, we used the fact that  $\nabla \times \vec{F} = (1, 0, 0)$  is perpendicular to  $\vec{N} = (0, 0, 1)$ , so the integrand is zero.