

Solutions to the Practice Problems
Math 210
October 18, 2005

1. Consider the function $f(x, y) = x^3y - yx^2 + 3xy$.

(a) Write down the equation of the tangent plane to the graph of f at $(0, 1)$.

Note $f(0, 1) = 0$. Also,

$$\frac{\partial f}{\partial x} = 3x^2y - 2xy + 3y, \quad \frac{\partial f}{\partial x}(0, 1) = 3, \quad \frac{\partial f}{\partial y} = x^3 - x^2 + 3x, \quad \frac{\partial f}{\partial y}(0, 1) = 0.$$

The tangent plane at $(0, 1)$ is given by

$$0 = (x - 0, y - 1, z - 0) \cdot \left(-\frac{\partial f}{\partial x}(0, 1), -\frac{\partial f}{\partial y}(0, 1), 1\right) = -3x + z,$$

which we can rearrange to read $z = 3x$.

(b) At which points (x, y) is the tangent plane horizontal?

We want to find points where the normal vector $(-\partial f/\partial x, -\partial f/\partial y, 1)$ is vertical, so we want $\partial f/\partial x = 0 = \partial f/\partial y$. Let's look at $\partial f/\partial x$ first:

$$0 = \frac{\partial f}{\partial x} = x^3 - x^2 + 3x = x(x^2 - x + 3) \Leftrightarrow x = 0.$$

Next we plug $x = 0$ into $\partial f/\partial x$ to see

$$0 = \frac{\partial f}{\partial x}(0, y) = 3y \Leftrightarrow y = 0.$$

Thus the only point with a horizontal tangent plane is $(0, 0)$.

2. Consider

$$f(x, y) = \int_x^y e^{t^2} dt.$$

(a) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

We use the Fundamental Theorem of Calculus:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_x^y e^{t^2} dt = -e^{x^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_x^y e^{t^2} dt = e^{y^2}.$$

(b) Does f have any critical points? Explain your answer.

Critical points satisfy $\partial f/\partial x = 0 = \partial f/\partial y$. However, both partial derivatives are exponentials, which are never zero. So there are no critical points.

3. Let some of the level sets of the function f be given by the figure below.

(a) Estimate $\frac{\partial f}{\partial x}(2, 0)$

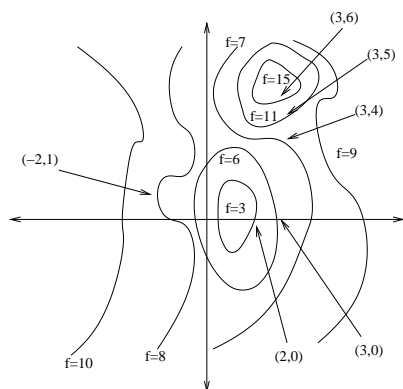
Notice that $(2, 0)$ is on the $f = 3$ level set, while $(3, 0)$ is on the $f = 6$ level set. So

$$\frac{\partial f}{\partial x}(2, 0) \approx \frac{6 - 3}{3 - 2} = 3.$$

(b) Estimate $\frac{\partial f}{\partial y}(3, 4)$

Notice that $(3, 4)$ is on the $f = 7$ level set, while $(3, 5)$ is on the $f = 11$ level set. So

$$\frac{\partial f}{\partial y}(3, 4) \approx \frac{11 - 7}{5 - 4} = 4.$$



(c) In which direction does $\nabla f(-2, 1)$ point?

Notice that $(-2, 1)$ is on the $f = 8$ level set, and this curve is vertical at $(-2, 1)$. Also, the $f = 10$ level set is to the left and the $f = 6$ and $f = 7$ level sets are to the right. Thus, because ∇f is perpendicular to level sets and points in the direction f increases, we know $\nabla f(-2, 1)$ points directly to the left (in the $(-1, 0)$ direction).

(d) Suppose you know $(1, 0)$ is a critical point. Would you guess it's a local maximum, a local minimum, or neither? Explain your answer.

You would guess that $(1, 0)$ is a local minimum because the function values of f are increasing as you move away from $(1, 0)$.

4. Consider $f(x, y) = xe^{x^2+y^2}$ and let $(x_0, y_0) = (1, 1)$.

(a) Compute $\nabla f(1, 1)$.

First note that

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (e^{x^2+y^2} + 2x^2e^{x^2+y^2}, 2xye^{x^2+y^2}) = e^{x^2+y^2}(1 + 2x^2, 2xy).$$

Evaluating this at $(1, 1)$, we get $\nabla f(1, 1) = (3e^2, 2e^2)$.

(b) Find the directional derivative $\nabla f \cdot \vec{u}(1, 1)$, where $\vec{u} = (1/2, \sqrt{3}/2)$.

$$\nabla f \cdot \vec{u}(1, 1) = (\nabla f(1, 1)) \cdot \vec{u} = (3e^2, 2e^2) \cdot (1/2, \sqrt{3}/2) = e^2(3/2 + \sqrt{3}).$$

(c) What is the direction of steepest increase for f , starting at $(1, 1)$. (Be sure to write down a unit vector!)

The direction of steepest increase is always $\nabla f/|\nabla f|$ (think about directional derivatives), which in this case is $(1/\sqrt{13})(3, 2)$.

(d) Notice $f(1, 1) = e^2$. What is the equation of the tangent line to the level set $\{f = e^2\}$, at the point $(1, 1)$?

Recall that ∇f is always perpendicular to level curves. Since the lines parallel to $\nabla f(1, 1)$ have slope $2/3$, this means the tangent to the level curve has slope $-3/2$. Thus the tangent line has the equation

$$y - 1 = -\frac{3}{2}(x - 1) \Leftrightarrow y = -\frac{3}{2}x + \frac{1}{2}.$$

5. Consider $f(x, y) = \cos x \cos y$.

(a) Classify all the critical points of f .

First we find the gradient and second partial derivatives of f :

$$\nabla f = (-\sin x \cos y, -\cos x \sin y), \quad \frac{\partial^2 f}{\partial x^2} = -\cos x \cos y = \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \sin x \sin y.$$

The critical points occur when $\nabla f = (0, 0)$, so we must have $\sin x \cos y = 0 = \cos x \sin y$. There are two possibilities: either $\sin x = 0$, which forces $\sin y = 0$, or $\cos y = 0$, which forces $\cos x = 0$. In the first case, we must have

$$(x, y) = (x_n, y_m) = (n\pi, m\pi)$$

for some integers n and m . In the second case we must have

$$(x, y) = (\hat{x}_n, \hat{y}_m) = \left(\frac{(2n+1)\pi}{2}, \frac{(2m+1)\pi}{2} \right).$$

Next we check the discriminant:

$$D = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 = \cos^2 x \cos^2 y - \sin^2 x \sin^2 y.$$

In the first case, we plug in $(x_n, y_m) = (n\pi, m\pi)$ to get $D = 1$ and $\partial^2 f / \partial x^2 = -\cos(n\pi) \cos(m\pi) = (-1)^{n+m+1}$. So this point is a local maximum for $n + m$ even and a local minimum for $n + m$ odd. In the second case, we plug in $(\hat{x}_n, \hat{y}_m) = ((2n + 1)\pi/2, (2m + 1)\pi/2)$ to get $D = -1$, so these points are all saddle points.

- (b) Find the absolute maximum and minimum of f on the square $\pi/4 \leq x \leq 3\pi/4$, $\pi/4 \leq y \leq 3\pi/4$.

Observe that the only critical point in this square is $(\pi/2, \pi/2)$, which is a saddle point. Thus the maximum of f must occur on the boundary of the square. Plug in $x = \pi/4$ to get

$$g_1(y) = f(\pi/4, y) = \frac{\cos y}{\sqrt{2}}, \quad \pi/4 \leq y \leq 3\pi/4.$$

The derivative of this function is $g_1' = -\sin(y)/\sqrt{2} < 0$ on $\pi/4 \leq y \leq 3\pi/4$. Thus g_1 is maximized at $y = \pi/4$, with a maximum value of $1/2$. Next plug in $x = 3\pi/4$ to get

$$g_2(y) = f(3\pi/4, y) = -\frac{\cos y}{\sqrt{2}}, \quad \pi/4 \leq y \leq 3\pi/4.$$

We can use symmetry of \cos . Indeed, $g_2(y) = -g_1(\pi - y)$, so the maximum value of $1/2$ occurs at $y = 3\pi/4$. Next we plug in $y = \pi/4$ to get

$$h_1(x) = f(x, \pi/4) = \frac{\cos x}{\sqrt{2}}, \quad \pi/4 \leq x \leq 3\pi/4.$$

The derivative of h_1 is $h_1' = -\sin(x)/\sqrt{2} < 0$ on $[\pi/4, 3\pi/4]$, so the maximum occurs at $x = \pi/4$, with a maximum value of $1/2$. (This is the same maximum we found earlier.) Finally, we plug in $y = 3\pi/4$ to get

$$h_2(x) = f(x, 3\pi/4) = -\frac{\cos x}{\sqrt{2}}, \quad \pi/4 \leq x \leq 3\pi/4.$$

Again, we use symmetry: $h_2(x) = -h_1(\pi - x)$, so the maximum of $1/2$ occurs at $x = 3\pi/4$. We conclude that the maximum value of f is $1/2$, which occurs at $(\pi/4, \pi/4)$ and $(3\pi/4, 3\pi/4)$. A similar analysis shows that the minimum value of $-1/2$ occurs at $(\pi/4, 3\pi/4)$ and $(3\pi/4, \pi/4)$.

6. Consider the composition $F(t) = f(x(t), y(t))$, where f is a function of the two variables x and y , while x and y are both functions of t .

- (a) Suppose $x(1) = 0$, $y(1) = 2$, $x'(1) = 2$, $y'(1) = -1$, $\frac{\partial f}{\partial x}(0, 2) = 5$, and $\frac{\partial f}{\partial y}(0, 2) = -3$. Compute $F'(1)$.

$$F'(1) = \frac{\partial f}{\partial x}(x(1), y(1))x'(1) + \frac{\partial f}{\partial y}(x(1), y(1))y'(1) = 5 \cdot 2 + (-3) \cdot (-1) = 13$$

- (b) If $x'(0) = 0$ and $y'(0) = 0$, is it true that $F'(0) = 0$? Explain your answer.

By the chain rule,

$$F' = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y'.$$

Thus if x' and y' are both zero then the sum is zero. So the answer is "yes."

- (c) If $F'(3) = 0$, is it true that $x'(3) = 0$ and $y'(3) = 0$? Explain your answer.

In this case, the answer is "no" because we can have cancellation. For instance, take $f(x, y) = x^2 - y^2$ and $(x(t), y(t)) = (t, t)$.

7. Compute the slope of the tangent line to the hyperbola $x^2 - y^2 = 1$ at the point $(2, \sqrt{3})$.

Notice that the hyperbola is given as a level set of the function $f(x, y) = x^2 - y^2$, so its tangents are perpendicular to ∇f . We compute that

$$\nabla f(2, \sqrt{3}) = (2x, -2y)|_{(2, \sqrt{3})} = (4, -2\sqrt{3}).$$

Thus the tangent to the hyperbola has slope $2/\sqrt{3}$.

8. (a) Evaluate $\int_0^1 \int_0^1 [x^2y + yx^3] dx dy$.

$$\int_0^1 \int_0^1 [x^2y + yx^3] dx dy = \int_0^1 \left[\frac{1}{3}x^3y + \frac{1}{4}x^4y \right]_{x=0}^{x=1} dy = \int_0^1 \frac{7}{12}y dy = \frac{7}{24}$$

- (b) Evaluate $\int_{x^2+y^2 \leq 1} [e^{x^2+y^2}] dx dy$. (Hint: try polar coordinates.)

In polar coordinates, this integral is

$$\int_0^{2\pi} \int_0^1 r e^{r^2} dr d\theta = \pi \int_0^1 e^u du = \pi(e-1).$$

- (c) Evaluate $\int_D [xy - x^2 y^2] dx dy$, where D is the ice-cream cone shaped region bounded by $y = x-1$, $y = -x-1$, and $x^2 + y^2 = 1$. (It might help to draw a picture of D).

The region D looks like an ice cream cone. Notice that D is symmetric about the y -axis while xy is anti-symmetric, which means

$$\int \int_D xy dA = 0.$$

Let D^+ be the part of D with $x \geq 0$. Then (because $x^2 y^2 = (-x)^2 y^2$), we have

$$\begin{aligned} \int \int_D xy - x^2 y^2 dA &= -2 \int \int_{D^+} x^2 y^2 dA \\ &= \int_0^1 \int_{x-1}^{\sqrt{1-x^2}} x^2 y^2 dy dx = -\frac{2}{3} \int_0^1 x^2 y^3 \Big|_{y=x-1}^{y=\sqrt{1-x^2}} dx \\ &= -\frac{2}{3} \int_0^1 x^2 (1-x^2)^{3/2} - x^2 (x-1)^3 dx. \end{aligned}$$

We first evaluate $\int_0^1 x^2 (x-1)^3 dx$:

$$\int_0^1 x^2 (x-1)^3 dx = \int_0^1 x^5 - 3x^4 + 3x^3 - x^2 dx = \frac{1}{6} - \frac{3}{5} + \frac{3}{4} - \frac{1}{3} = -\frac{1}{60}.$$

Next we evaluate $\int_0^1 x^2 (1-x^2)^{3/2} dx$. We will first integrate by parts (with $u = x$ and $dv = x(1-x^2)^{3/2} dx$) and then do a trigonometric substitution (with $x = \sin \theta$).

$$\begin{aligned} \int_0^1 x^2 (1-x^2)^{3/2} dx &= -\frac{x}{5} (1-x^2)^{3/2} \Big|_0^1 + \frac{1}{5} \int_0^1 (1-x^2)^{5/2} dx = \frac{1}{5} \int_0^1 (1-x^2)^{5/2} dx \\ &= \frac{1}{5} \int_{\pi/2}^0 (1-\sin^2 \theta)^{5/2} (\cos \theta) d\theta = \int_0^{\pi/2} \cos^6 \theta d\theta = \frac{1}{5} \int_0^{\pi/2} ((1/2)(1+\cos(2\theta)))^3 d\theta \\ &= \frac{1}{40} \int_0^{\pi/2} 1 + 3\cos(2\theta) + 3\cos^2(2\theta) + 3\cos^3(2\theta) d\theta \\ &= \frac{\pi}{80} + \frac{3}{80} \sin(2\theta) \Big|_0^{\pi/2} + \frac{3}{40} \int_0^{\pi/2} (1/2)(1+\cos(4\theta)) d\theta + \frac{1}{40} \int_0^{\pi/2} \cos(2\theta)(1-\sin^2(2\theta)) d\theta \\ &= \frac{\pi}{80} + \frac{3\pi}{160} + \frac{3}{80} \int_0^1 (1-u^2) du = \frac{\pi}{80} + \frac{3\pi}{160} + \frac{1}{40} = \frac{5\pi+4}{160}. \end{aligned}$$

Putting this all together, we have

$$\int \int_D xy - x^2 y^2 dA = -\frac{2}{3} \cdot \frac{5\pi+4}{160} + \frac{2}{3} \cdot \frac{-1}{60} = \frac{4-5\pi}{240} - \frac{1}{90} = \frac{4-15\pi}{720}.$$

9. Find the minimum of $f = y^3 - 3yx^2$ on the ellipse $x^2 + 4y^2 \leq 1$.

We first find interior critical points:

$$(0,0) = \nabla f = (-6xy, 3y^2 - 3x^2) \Leftrightarrow (x,y) = (0,0).$$

However, one can see that $(0,0)$ is neither a local max nor a local min. (The easiest way to see this is to look at the zero set. This divides the plane into six wedges all meeting at $(0,0)$, and the sign of f alternates as one travels from wedge to wedge.) So we look for critical points on the boundary using Lagrange multipliers. The boundary is given by $g(x,y) = x^2 + 4y^2 = 1$, so we set

$$\nabla f = \lambda \nabla g, \quad g = 1,$$

which is the system of equations

$$-6xy = 2\lambda x, \quad 3y^2 - 3x^2 = 8\lambda y, \quad x^2 + 4y^2 = 1.$$

If $x \neq 0$, the first equation gives $\lambda = -3y$. Plugging this into the second equation, we get $3y^2 - 3x^2 = -24y^2$, which we rearrange as $-x^2 + 9y^2 = 0$. Combine this with the last equation, we get $13y^2 = 1$, or $y = \pm 1/\sqrt{13}$, which also gives us $x = \pm 3/\sqrt{13}$. Finally, if $x = 0$ we must have $y = \pm 1/2$. This gives us 6 points to check:

$$\begin{aligned} f(0, 1/2) &= 1/8, & f(0, -1/2) &= -1/8, \\ f(3/\sqrt{13}, 1/\sqrt{13}) &= -26/\sqrt{169}, & f(3/\sqrt{13}, -1/\sqrt{13}) &= 26/\sqrt{169}, \\ f(-3\sqrt{13}, 1\sqrt{13}) &= -26/\sqrt{169}, & f(-3\sqrt{13}, -1\sqrt{13}) &= 26/\sqrt{169}. \end{aligned}$$

So we see that the minimum value is $-26/\sqrt{169}$, which occurs at $(3/\sqrt{13}, 1/\sqrt{13})$ and $(-3/\sqrt{13}, 1/\sqrt{13})$.

10. Consider the function

$$f(x, y) = \int_{2-y}^x \sqrt{1+t^2} dt.$$

(a) (5 points) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Use the fundamental theorem of calculus for functions of one variable:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{2-y}^x \sqrt{1+t^2} dt = \sqrt{1+x^2}$$

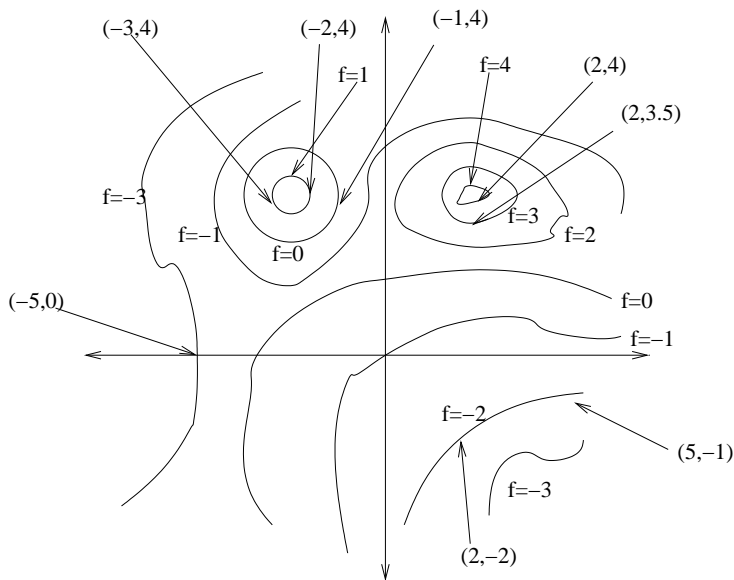
and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_{2-y}^x \sqrt{1+t^2} dt = -\sqrt{1+(2-y)^2}(-1) = \sqrt{1+(2-y)^2}$$

(b) (5 points) Does f have any critical points? Be sure to explain your answer.

No. Both partial derivative are always positive, so there can never be a point (x_0, y_0) with $\nabla f(x_0, y_0) = (0, 0)$.

11. Consider the following sketch of level curves of the function $f(x, y)$.



(a) (3 points) Estimate $\frac{\partial f}{\partial y}(2, 3.5)$.

$$\frac{\partial f}{\partial y}(2, 3.5) \approx \frac{f(2, 4) - f(2, 3.5)}{4 - 3.5} = 2$$

(b) (4 points) In which direction does $\nabla f(-5, 0)$ point? Be sure to explain your answer.

First observe that $(-5, 0)$ lies on the $f = -3$ level curve, so $\nabla f(-5, 0)$ must be perpendicular to that curve. At $(-5, 0)$, this level curve is vertical, so ∇f points either in the $(1, 0)$ direction or in the $(-1, 0)$ direction (to the right or the left). Also, f increases if you move to the right, while it decreases if you move to the left. Thus $\nabla f(-5, 0)$ points in the $(1, 0)$ direction (to the right).

- (c) (3 points) If f has a critical point at $(-2.5, 4.5)$, do you expect it to be a local minimum, a local maximum, or a saddle point? Be sure to explain your answer.

Because f decreases as you move away from $(-2.5, 4.5)$ in any direction, you would expect that f has a local maximum there.

12. Consider the function

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

- (a) (5 points) Verify that the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Note that

$$\nabla f = (4x^3 - 4y, 4y^3 - 4x).$$

Critical points satisfy $\nabla f = (0, 0)$, which we can rewrite as

$$x^3 = y, \quad y^3 = x.$$

The substituting the first equation in to the second, we get

$$x^9 = x \Leftrightarrow 0 = x^9 - x = x(x^8 - 1).$$

The only solutions to this equation are $x = 0, 1, -1$. If $x = 0$ we get $y = 0$, if $x = 1$ we get $y = 1$, and if $x = -1$ we get $y = -1$. Thus $(0, 0)$, $(1, 1)$ and $(-1, -1)$ are the only critical points.

- (b) (5 points) Classify these critical points as local maxima, local minima, or saddle points.

First we compute second derivatives and the discriminant:

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \quad \frac{\partial^2 f}{\partial y^2} = 12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = -4, \quad D = 144x^2y^2 - 16.$$

Plugging in our critical points, we see $D(0, 0) = -16 < 0$, so $(0, 0)$ is a saddle point. Next, $D(1, 1) = 140 > 0$ and $\partial^2 f / \partial x^2(1, 1) = 12 > 0$, so $(1, 1)$ is a local minimum. Finally, $D(-1, -1) = 140 > 0$ and $\partial^2 f / \partial x^2 = 12 > 0$, so $(-1, -1)$ is also a local minimum.

13. (10 points) Find the absolute maximum of $f(x, y) = xy$ on the ellipse $g(x, y) = x^2 + 4y^2 \leq 1$.

First we look for interior critical points:

$$(0, 0) = \nabla f = (y, x) \Leftrightarrow x = 0 = y.$$

However, one can check that $D \equiv -1$ in this case, so $(0, 0)$ is a saddle point and the maximum must occur on the boundary. Next we use Lagrange multipliers:

$$\nabla f = \lambda \nabla g, \quad g = 1,$$

which we rewrite as

$$y = 2\lambda x, \quad x = 8\lambda y, \quad x^2 + 4y^2 = 1.$$

Then either $x = 0$ (which implies $y = 0$, a case we've already examined), or $16\lambda^2 = 1$. We plug $\lambda = \pm 1/4$ back into our equations to get $x = \pm 2y$, so

$$1 = x^2 + 4y^2 = 4y^2 + 4y^2 \Leftrightarrow y = \pm \frac{1}{\sqrt{8}}.$$

This in turn gives $x = \pm 1/\sqrt{2}$, and so we have four candidates for maximum points:

$$(1/\sqrt{2}, \sqrt{2}), (-1/\sqrt{2}, \sqrt{2}), (1/\sqrt{2}, -\sqrt{2}), (-1/\sqrt{2}, -\sqrt{2}).$$

Testing these, we find maximal function values (of $1/4$) at $\pm(1/\sqrt{2}, 1/\sqrt{2})$. The other two points are minima.

14. (a) (5 points) Where D is the square $\{1 \leq x \leq 2, -2 \leq y \leq -1\}$, evaluate

$$\int \int_D [x^2y + yx^3] dA.$$

We'll integrate with respect to x first, but in this case it doesn't make too much difference.

$$\begin{aligned} \int \int_D [x^2y + yx^3] dA &= \int_{-2}^{-1} \int_1^2 [x^2y + yx^3] dx dy = \int_{-2}^{-1} \left[\frac{1}{3}x^3y \Big|_1^2 + \frac{1}{4}x^4y \Big|_1^2 \right] dy \\ &= \int_{-2}^{-1} [8y/3 + 15y/4] dy = \frac{77}{12} \int_{-2}^{-1} y dy = \frac{77}{24} y^2 \Big|_{-2}^{-1} = -\frac{77}{8}. \end{aligned}$$

- (b) (5 points) Set up, but do **not** evaluate, the integral $\int \int_D \sqrt{1+x^2+y^2} dA$, where D is the domain bounded by the curves $y = x + 1$ and $x = -y^2$.

We'll integrate first with respect to x (taking horizontal slices), because it's much easier. It helps to draw the domain. It's a region bounded on the left by a slanted line and on the right by a parabola opening to the left. One can find the intersection points of these two curves by setting

$$-y^2 = x = y - 1 \Leftrightarrow 0 = y^2 + y - 1 \Leftrightarrow y = \frac{-1 \pm \sqrt{5}}{2}.$$

This gives the upper and lower limits of the y -integration. For a given y , the lower limit in the x -integral is the left-most point, which is $x = y - 1$, and the upper is the right-most point, which is $x = -y^2$. Thus we have

$$\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} \int_{y-1}^{-y^2} \sqrt{1+x^2+y^2} dx dy.$$

15. Consider the function

$$f(x, y) = \int_{1-y}^{x^2} \frac{dt}{1+t^2}.$$

- (a) (4 points) Compute the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$.

Use the fundamental theorem of calculus and the chain rule.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{1-y}^{x^2} \frac{dt}{1+t^2} = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_{1-y}^{x^2} \frac{dt}{1+t^2} = -\frac{1}{1+(1-y)^2} \cdot (-1) = \frac{1}{1+(1-y)^2}$$

- (b) (4 points) Does f have any critical points? Be sure to explain your answer.

No, $\partial f / \partial y$ is never zero.

16. Consider the function

$$f(x, y) = x^2 + y^3 - 6xy.$$

- (a) (3 points) Verify that the only critical points of f are $(0, 0)$ and $(18, 6)$.

First we compute the gradient:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x - 6y, 3y^2 - 6x).$$

Critical points occur when $\nabla f = (0, 0)$. Setting $\partial f / \partial x = 0$, we get $x = 3y$. Now plug this into $\partial f / \partial y = 0$:

$$0 = \frac{\partial f}{\partial y} = 3y^2 - 6x = 3y^2 - 18y \Leftrightarrow y = 0, 6.$$

If $y = 0$ the $x = 0$, giving us the critical point $(0, 0)$. If $y = 6$ then $x = 18$, giving us the critical point $(18, 6)$.

- (b) (2 points) Classify each of these two critical points as local minima, local maxima, or saddle points.

First find the second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = -6, \quad \frac{\partial^2 f}{\partial y^2} = 6y.$$

In the case of $(0, 0)$, the discriminant is -36 , so $(0, 0)$ is a saddle point. In the case of $(18, 6)$, the discriminant is 36 and $\partial^2 f / \partial x^2 = 2$, so $(18, 6)$ is a local minimum.

- (c) (3 points) Observe that $f(1, 1) = -4$. Find the equation of the tangent line to the level set $\{f = -4\}$ at the point $(1, 1)$.

Recall that the gradient is perpendicular to level sets. We have

$$\nabla f(1, 1) = (2x - 6y, 3y^2 - 6x)|_{(1,1)} = (-4, -3),$$

so the line in the perpendicular direction has slope $-4/3$. Thus the equation of the tangent line is

$$(y - 1) = -\frac{4}{3}(x - 1) \Leftrightarrow y = -\frac{4}{3}x + \frac{7}{3}.$$

(d) (2 points) Compute $D_{\vec{u}}f(1, 1)$, where $\vec{u} = (\sqrt{3}/2, 1/2)$.

$$D_{\vec{u}}f(1, 1) = \nabla f(1, 1) \cdot \vec{u} = (-4, -3) \cdot (\sqrt{3}/2, 1/2) = -\frac{4\sqrt{3} + 3}{2}.$$

17. Let $F = F(u, v)$, while $u = u(t)$ and $v = v(t)$. Call the resulting composition $f(t) = F(u(t), v(t))$.

(a) (2 points) If $u(0) = 1$, $u'(0) = 2$, $v(0) = -1$, $v'(0) = 3$ and $\nabla F(1, -1) = (8, -1)$, then what is $f'(0)$?

$$f'(0) = \left. \frac{\partial F}{\partial u} \right|_{(u(0), v(0))} u'(0) + \left. \frac{\partial F}{\partial v} \right|_{(u(0), v(0))} v'(0) = 8 \cdot 2 + (-1) \cdot 3 = 13.$$

(b) (3 points) If $u'(1) = 0$ and $v'(1) = 0$ then is it always true that $f'(1) = 0$? Be sure to explain your answer.
Yes. By the chain rule, $f'(1) = \nabla F \cdot (u'(1), v'(1)) = \nabla F \cdot (0, 0) = 0 + 0 = 0$.

(c) (3 points) If $f'(-1) = 0$, is it always true that $u'(-1) = 0 = v'(-1)$? Be sure to explain your answer.
No. It is possible that $\nabla F(u(-1), v(-1)) = (0, 0)$, which would mean that $f'(-1) = 0$. Or you could get cancellation: a positive term and a negative term adding up to give you zero.

18. (8 points) Find the maximum and minimum of $f(x, y) = xy$ subject to the constraint $x^4 + y^4 = 1$.

Use Lagrange multipliers: $\nabla f = \lambda \nabla g$ for some nonzero constant λ :

$$y = 4\lambda x^3, \quad x = 4\lambda y^3, \quad x^4 + y^4 = 1.$$

Notice that if $x = 0$ then $y = 0$ (and vice versa), which does not satisfy the constraint $x^4 + y^4 = 1$. Solve for λ in the first two equations:

$$\frac{y}{4x^3} = \lambda = \frac{x}{4y^3} \Leftrightarrow x^4 = y^4 \Leftrightarrow x = \pm y.$$

Plug this into the constraint:

$$1 = x^4 + y^4 = 2x^4 \Leftrightarrow x = \pm 2^{-1/4}, y = \pm 2^{-1/4},$$

which gives four points (all possible sign combinations). Plugging these points in to f , we see that $(2^{-1/4}, 2^{-1/4})$ and $(-2^{-1/4}, -2^{-1/4})$ are a maximum points ($f = 2^{-1/2}$) while $(2^{-1/4}, -2^{-1/4})$ and $(-2^{-1/4}, 2^{-1/4})$ are minimum points ($f = -2^{-1/2}$).

19. (a) (4 points) Evaluate $\int \int_D [x^2 y] dA$ where $D = [1, 2] \times [2, 3]$.

$$\begin{aligned} \int \int_D x^2 y dA &= \int_1^2 \int_2^3 x^2 y dy dx = \int_1^2 \left. \frac{1}{2} x^2 y^2 \right|_{y=2}^{y=3} \\ &= \frac{5}{2} \int_1^2 x^2 dx = \frac{5}{6} x^3 \Big|_1^2 = \frac{35}{6} \end{aligned}$$

(b) (4 points) Set up, but **do not** evaluate $\int \int_D e^{1-x^2-y^2} dA$, where D is bounded by $x = y^2 - y$ and $y = x/2$. (Hint: it might help to draw a picture.)

The curve $x = y^2 - y$ is a parabola opening to the right, passing through the y -axis at $y = 0$ and $y = 1$. The curve $y = x/2$ is a line through the origin with slope $1/2$. The two intersect at

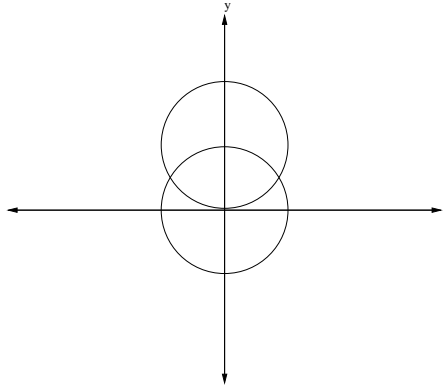
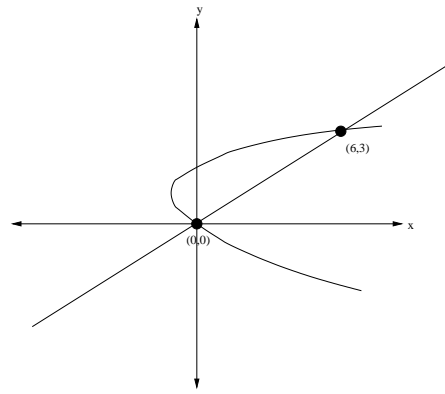
$$2y = y^2 - y \Leftrightarrow 0 = y^2 - 3y = y(y - 3) \Leftrightarrow y = 0, 3.$$

Notice that the line lies to the right of the parabola in the area we're interested in ($0 \leq y \leq 3$).

We choose horizontal slices for this integral, because for vertical slices we'd have to write down two integrals:

$$\int \int_D e^{1-x^2-y^2} dA = \int_0^3 \int_{y^2-y}^{2y} e^{1-x^2-y^2} dx dy.$$

20. Consider the two circle given in polar coordinates coordinates by $\{r = 2\}$ and $\{r = 4 \sin \theta\}$.



- (a) (4 points) Sketch these two circles and find their points of intersection.

Both circles have radius 2. In cartesian coordinates, the center of the circle $\{r = 2\}$ is $(0, 0)$, while the center of the circle $\{r = 4 \sin \theta\}$ is $(0, 2)$. The two circles intersect where

$$2 = 4 \sin \theta \Leftrightarrow \sin \theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

(And, of course, $r = 2$.) In cartesian coordinates, this gives the points $(\pm\sqrt{3}, 1)$.

- (b) (4 points) Find the area lying inside the circle $\{r = 4 \sin \theta\}$ but outside the circle $\{r = 2\}$. (Hint: the area is given by a double integral. You might find some of the trigonometric identities on the front useful.)

We write the integral $\iint_D dA$ in polar coordinates:

$$\begin{aligned} \iint_D dA &= \int_{\pi/6}^{5\pi/6} \int_2^{4 \sin \theta} r dr d\theta = \frac{1}{2} \int_{\pi/6}^{5\pi/6} r^2 \Big|_{r=2}^{r=4 \sin \theta} d\theta \\ &= \int_{\pi/6}^{5\pi/6} 8 \sin^2 \theta - 2 d\theta = \int_{\pi/6}^{5\pi/6} 4(1 - \cos(2\theta)) - 2 d\theta = \int_{\pi/6}^{5\pi/6} 2 - 4 \cos(2\theta) d\theta \\ &= \frac{4\pi}{3} - 2 \sin(2\theta) \Big|_{\pi/6}^{5\pi/6} = \frac{4\pi}{3} - 2\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} + 2\sqrt{3}. \end{aligned}$$