Solutions to the Midterm Exam

Math 210

October 31, 2006

1. Consider the function

$$f(x,y) = \int_{x^2}^{e^y} \ln(1+t)dt.$$

   (a) (4 points) Compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$.

   Use the fundamental theorem of calculus:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_{x^2}^{e^y} \ln(1+t)dt = -\ln(1+x^2)(x^2)' = -2x\ln(1+x^2),$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_{x^2}^{e^y} \ln(1+t)dt = \ln(1+e^y)(e^y)' = e^y\ln(1+e^y),$$

   (b) (4 points) Does f have any critical points? Be sure to explain your answer.

   No. Notice that $e^y > 0$, so $\ln(1+e^y) > \ln(1) = 0$, which means $\partial f/\partial y > 0$.

2. Consider the function

$$f(x,y) = x^2y - 2x^2 - xy + 2x.$$

   (a) (3 points) Verify that the only critical points of f are $(0,2)$ and $(1,2)$.

   We first take the partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy - 4x - y + 2, \qquad \frac{\partial f}{\partial y} = x^2 - x.$$

   Setting $\partial f/\partial y = 0$, we get the equation $0 = x^2 - x$, which only has solutions $x = 0, 1$. If we plug $x = 0$ into $\partial f/\partial x$, we get $0 = -y+2$, which forces $y = 2$. If we plug $x = 1$ into $\partial f/\partial x$, we get $0 = 2y-4-y+2$, chich again forces $y = 2$. Thus the only critical points are $(0,2)$ and $(1,2)$.

   (b) (2 points) Classify each of these two critical points as local minima, local maxima, or saddle points.

   We first evaluate second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 2y - 4, \qquad \frac{\partial^2 f}{\partial y^2} = 0, \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x - 1,$$

   and so the discriminant is

$$D = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial x \partial y})^2 = -(2x-1)^2.$$

   Plugging in our critical points, we have $D(0,2) = -(-1)^2 < 0$, and so $(0,2)$ is a saddle point. Simialarly, $D(1,2) = -1 < 0$, and so $(1,2)$ is also a saddle point.

   (c) (2 points) Compute $\nabla_{\vec{u}} f(1,1)$ where $\vec{u} = (1/2, \sqrt{3}/2)$.

   We compute $\nabla f(1,1) = (1-2-1+2, 0) = (-1, 0)$. Then the directional derivative is

$$\nabla_{\vec{u}} f(1,1) = (1/2, \sqrt{3}2) \cdot (-1, 0) = -1/2.$$

   (d) (2 points) Observe that $f(1,1) = 0$. Find the equation of the tangent line to the $\{f = 0\}$ level set at $(1,1)$.

   Recall that level sets are perpendicular to the gradient. In this case, the gradient is horizontal, so the tangent line to the level set is vertical. We want to write downs a vertical line through $(1,1)$, which is the line $x = 1$.

3. Let $F = F(u,v)$, while $u = u(t)$ and $v = v(t)$. Call the resulting composition $f(t) = F(u(t), v(t))$.

   (a) (2 points) If $u(0) = 1$, $u'(0) = 2$, $v(0) = -1$, $v'(0) = 3$ and $\nabla F(1,-1) = (8,-1)$, then what is $f'(0)$?

$$f'(0) = \frac{\partial F}{\partial u}u'(0) + \frac{\partial F}{\partial v}v'(0) = 8 \cdot 2 + (-1) \cdot (3) = 13.$$

(b) (3 points) If $u'(1) = 0$ and $v'(1) = 0$ then is it always true that $f'(1) = 0$? Be sure to explain your answer.
Yes:

$$f'(1) = \frac{\partial F}{\partial u}u'(1) + \frac{\partial F}{\partial y}y'(1) = \frac{\partial F}{\partial u} \cdot (0) + \frac{\partial F}{\partial v} \cdot (0) = 0.$$

(c) (3 points) If $f'(-1) = 0$, is it always true that $u'(-1) = 0 = v'(-1)$? Be sure to explain your answer.
No. You could have, for instance, $\frac{\partial F}{\partial u}u' = 1$, and $\frac{\partial F}{\partial v}v' = -1$, then by the chain rul the derivative $f' = 1 - 1 = 0$.

4. (8 points) Find the maximum and minimum of $f(x, y) = x^3 + 3xy^2$ subject to the constraint $x^2/4 + y^2 = 1$.

There are two ways to do this problem.

Method 1: Lagrange multipliers. Let $g(x, y) = x^2/4 + y^2$, then use

$$(3x^2 + 3y^2, 6xy) = \nabla f = \lambda \nabla g = \lambda(x/2, 2y).$$

The seond equation gives us $6xy = 2\lambda y$. One possibility is $y = 0$, but if $y \neq 0$ then $\lambda = 3x$. Plug $\lambda = 3x$ into the first equation to get

$$3x^2 + 3y^2 = \frac{3}{2}x^2 \Leftrightarrow 0 = \frac{3}{2}x^2 + 3y^2.$$

The only solution to this equation is $x = 0, y = 0$, which doesn't satisfy the constraint, and so the max/min must occur when $y = 0$, which means $x = \pm 2$. Plugging $(\pm 2, 0)$ into f, we have $f(2, 0) = 8$ and $f(-2, 0) = -8$, and so the mazimal value is 8, achieved at $(2, 0)$ while the minimal value is -8, achieved at $(-2, 0)$.

Method 2: plug in $y^2 = 1 - x^2/4$ into the original function f, so we have $f = x^3 + 3x(1 - x^2/4) = x^3/4 + 3x$. The derivative of this restricted f is $f' = (3/4)x^2 + 3 > 0$, which means the restricted function is increasing in x. This tells us the function is largest when x is largest, at $x = 2, y = 0$ and the function is least when x is least, at $x = -2, y = 0$. The maximum is $f = 8$ and the minimum is $f = -8$.

5. (a) (4 points) Evaluate $\int\int_D [x^2 + 3xy]dA$ where $D = [1, 2] \times [0, 1]$.

$$\int\int_D [x^2 + 3xy]dA = \int_1^2 \int_0^1 [x^2 + 3xy]dydx = \int_1^2 [x^2 + (3/2)x]dx = (x^3/3)\Big|_1^2 + (3x^2/4)\Big|_1^2 = \frac{7}{3} + \frac{9}{4} = \frac{55}{12}$$

(b) (4 points) Set up, but **do not** evaluate $\int\int_D e^{1-x^2-y^2} dA$, where D is bounded by $y = x^2 - x$ and $x = 2y$. (Hint: it might help to draw a picture.)
The parabola $y = x^2 - x$ opens up, and passes through the x–axis at $x = 0, 1$. The line and parabola intesect at

$$\frac{x}{2} = x^2 - x \Leftrightarrow 0 = x^2 - \frac{3}{2}x \Leftrightarrow x = 0, \frac{3}{2}.$$

Also notice that the line lies above the parabola for $0 \leq x \leq 3/2$. We choose vertical slices, so that we can write the integral as

$$\int\int_D e^{1-x^2-y^2} dA = \int_0^{3/2} \int_{x^2-x}^{x/2} e^{1-x^2-y^2} dydx.$$

6. (a) (3 points) Let c_1 be the circle $(x - 1)^2 + y^2 = 1$. Rewrite the equation for c_1 in polar coordinates.
We use $x = r\cos\theta$ and $y = r\sin\theta$:

$$1 = (x - 1)^2 + y^2 = (r\cos\theta - 1)^2 + r^2\sin^2\theta = r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta = r^2 - 2r\cos\theta + 1.$$

which means $0 = r^2 - 2r\cos\theta$, or $r = 2\cos\theta$.

(b) (3 points) Let c_2 be the circle $x^2 + y^2 = 1$. Find the intersection points of c_1 and c_2. (There are two intersection points.)
The circle c_2 is given by $r = 1$. Equate $r = 1 = 2\cos\theta$ to get $\cos\theta = 1/2$, and so $\theta = \pi/3, 5\pi/3$ (or $\theta = \pm\pi/3$). Alternatively, we can use Cartesian coordinates:

$$x^2 + y^2 = (x - 1)^2 + y^2 \Leftrightarrow 0 = -2x + 1 \Leftrightarrow x = 1/2.$$

The plug $x = 1/2$ into the equation for either circle to get $y = \pm\sqrt{3}/2$.

(c) (3 points) Find the area of the region inside both circles c_1 and c_2. (Hint: the area is given by a double integral. You might find some of the trigonometric identities on the front useful.)

This one is a little tricky. It turns out that for $-\pi/3 \leq \theta \leq \pi/3$, the arc of the circle $r = 2\cos\theta$ is outside the arc of the circle $r = 1$, so it's easier to find the complementary area. If A is the area of the overlap, then

$$\begin{aligned}
\pi - A &= \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 - 1 d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos^2\theta - 1 d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2 + 2\cos(2\theta) - 1 d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} d\theta + \int_{-\pi/3}^{\pi/3} \cos(2\theta) d\theta = \frac{\pi}{3} + \frac{1}{2}\sin(2\theta)\Big|_{-\pi/3}^{\pi/3} \\
&= \frac{\pi}{3} + \frac{\sqrt{3}}{2}.
\end{aligned}$$

Putting this all together, we see $A = (2\pi)/3 - \sqrt{3}/2$.