Solutions to the First Midterm Exam

Math 210

Oct. 1, 2006

1. Consider the two vectors
$$\vec{v} = (-1, 2, 1), \qquad \vec{u} = (2, 0, 1).$$

   (a) (4 points) Find the cosine of the angle θ between $\vec{v}$ and $\vec{u}$.
$$\cos(\theta) = \frac{\vec{v} \cdot \vec{u}}{|\vec{v}||\vec{u}|} = \frac{(-1, 2, 1) \cdot (2, 0, 1)}{\sqrt{6}\sqrt{5}} = -\frac{1}{\sqrt{30}}.$$

   (b) (4 points) Find a vector $\vec{w}$ which is perpendicular to $\vec{v}$, and verify that $\vec{v} \perp \vec{w}$. (Hint: you don't need to take a cross product.)
   Recall that $\vec{w} \perp \vec{v}$ is and only if $\vec{w} \cdot \vec{v} = 0$, so we can choose $\vec{w} = (2, 1, 0)$ (among many other choices). To check:
$$\vec{w} \cdot \vec{v} = (2, 1, 0) \cdot (-1, 2, 1) = -2 + 2 + 0 = 0.$$

2. Consider the lanes Π_1 and Π_2, where Π_1 is given by the equation $x + z = 2$, while Π_2 contains the point $(1, 2, 1)$ and has the normal vector $\vec{n_2} = (0, 1, 1)$.

   (a) (4 points) Find the equation for Π_2.
   The equation for Π_2 is
$$y + z = (0, 1, 1) \cdot (1, 2, 1) = 3.$$

   (b) (4 points) Find a vector $\vec{v}$ which is parallel to both Π_1 and Π_2.
   We want $\vec{v}$ to be parallel to Π_1 and Π_2, so it must be perpendicular to $\vec{n_1}$ and $\vec{n_2}$ (here $\vec{n_1}$ is the normal to Π_1). Thus we can choose
$$\vec{v} = \vec{n_1} \times \vec{n_2} = (1, 0, 1) \times (0, 1, 1) = (-1, -1, 1).$$

   (c) (4 points) Parameterize the line l of intersection of Π_1 and Π_2.
   To parameterize the line l, we need a basepoint and a direction. The direction vector is $\vec{v} = (-1, -1, 1)$ we just found in the previous part of the problem. The basepoint is given by a simultaneous solution of the two equations
$$x + z = 2 \qquad y + z = 3.$$
   Because the third component of $\vec{v}$ is nonzero, we can choose $z = 0$ (the amounts to saying the line l hits the xy plane), so the equations above give $x = 2, y = 3$. Thus a parameterization for l is
$$l(t) = (2, 3, 0) + t(-1, -1, 1) = (2 - t, 3 - t, t).$$

3. Consider the parameterized curve
$$\vec{r}(t) = (\cos(t), t, \sin(2t)).$$

   (a) (4 points) Find the velocity vector $\frac{d\vec{r}}{dt}(\pi/2)$ at time $t = \pi/2$.
$$\frac{d\vec{r}}{dt}(\pi/2) = (-\sin t, 1, 2\cos(2t))|_{t=\pi/2} = (-1, 1, -2).$$

   (b) (4 points) Is the line tangent to $\vec{r}$ ever parallel to the xz plane? Be sure to explain your answer.
   Notice that the normal to the xz plane is $\vec{n} = (0, 1, 0)$. Thus the tangent line will be parallel to the plane precisely when it is perpendicular to $\vec{n}$. However, taking a dot product we see
$$\vec{n} \cdot \frac{d\vec{r}}{dt} = (0, 1, 0) \cdot (-\sin(t), 1, 2\cos(2t)) = 1 \neq 0,$$
   so the tangent line will never be parallel to the xz plane.

   (c) (4 points) Set up, but do not evaluate, the arclength integral for $0 \leq t \leq 2\pi$.
   Call the length L:
$$L = \int_0^{2\pi} |\frac{d\vec{r}}{dt}|dt = \int_0^{2\pi} |(-\sin(t), 1, 2\cos(2t))|dt = \int_0^{2\pi} \sqrt{1 + \sin^2 t + 4\cos^2(2t)}dt.$$

4. Consider the function $f(x, y) = \ln(1 + x^2 + y^2)$.

   (a) (5 points) Describe and sketch the $\{f = 2\}$ level set.
   All points (x, y) in the $\{f = 2\}$ level set satisfy

   $$2 = f(x, y) = \ln(1 + x^2 + y^2) \Leftrightarrow e^2 = 1 + x^2 + y^2 \Leftrightarrow e^2 - 1 = x^2 + y^2.$$

   This describes a circle centered at the origin with radius $\sqrt{e^2 - 1}$.

   (b) (4 points) For which values of z does the level set $\{f = z\}$ not contain any points? Be sure to explain your answer.
   Notice that (i) $1 + x^2 + y^2 \geq 1$ and (ii) ln is an increasing function. Putting these two facts together we see

   $$0 = \ln(1) \leq \ln(1 + x^2 + y^2) = f(z),$$

   so $\{f = z\}$ is empty precisely when $z < 0$.

5. Consider the function $f(x, y) = x \sin(1 + x - y) + y \ln(1 + x^2)$.

   (a) (5 points) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

   $$\frac{\partial f}{\partial x} = \sin(1 + x - y) + x \cos(1 + x - y) + \frac{2xy}{1 + x^2}, \quad \frac{\partial f}{\partial y} = -x \cos(1 + x - y) + \ln(1 + x^2).$$

   (b) (4 points) Is the tangent plane to the graph of f ever parallel to the plane $x = -y$? Be sure to explain your answer.
   No this is never possible. Recall that the tangent plane to the graph has the normal $\vec{n} = (-\partial f/\partial x, -\partial f, \partial y, 1)$, while the normal to the $x = -y$ plane is $\vec{N} = (1, 1, 0)$. Because the last component for $\vec{N}$ is zero while the last component of $\vec{n}$ is nonzero, there is no nonzero constant λ such that $\vec{N} = \lambda \vec{n}$. In other words, $\vec{n}$ and $\vec{N}$ are not parallel.