Solutions to the Midterm Exam

Math 210

March 28, 2005

1. Consider the function

$$f(x,y) = \int_{2-y}^{x} \sqrt{1+t^2}dt.$$

   (a) (5 points) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

   Use the fundamental theorem of calculus for functions of one variable:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_{2-y}^{x} \sqrt{1+t^2}dt = \sqrt{1+x^2}$$

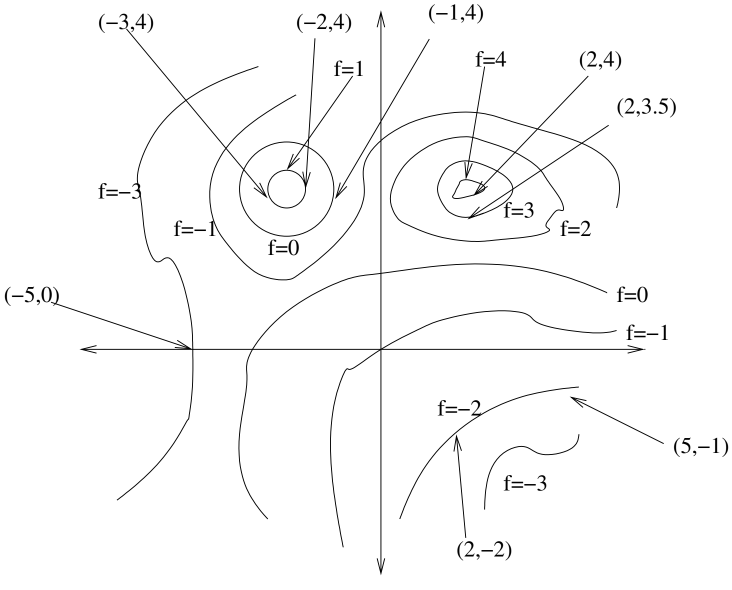
   and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_{2-y}^{x} \sqrt{1+t^2}dt = -\sqrt{1+(2-y)^2}(-1) = \sqrt{1+(2-y)^2}$$

   (b) (5 points) Does f have any critical points? Be sure to explain your answer.

   No. Both partial derivative are always positive, so there can never be a point (x_0, y_0) with $\nabla f(x_0, y_0) = (0,0)$.

2. Consider the following sketch of level curves of the function $f(x,y)$.



   (a) (3 points) Estimate $\frac{\partial f}{\partial y}(2, 3.5)$.

$$\frac{\partial f}{\partial y}(2,3.5) \approx \frac{f(2,4) - f(2,3.5)}{4-3.5} = 2$$

   (b) (4 points) In which direction does $\nabla f(-5,0)$ point? Be sure to explain your answer.

   First observe that $(-5,0)$ lies on the $f = -3$ level curve, so $\nabla f(-5,0)$ must be perpendicular to that curve. At $(-5,0)$, this level curve is vertical, so ∇f points either in the $(1,0)$ direction or in the $(-1,0)$ direction (to the right or the left). Also, f increases if you move to the right, while it decreases if you move to the left. Thus $\nabla f(-5,0)$ points in the $(1,0)$ direction (to the right).

   (c) (3 points) If f has a critical point at $(-2.5, 4.5)$, do you expect it to be a local minimum, a local maximum, or a saddle point? Be sure to explain your answer.

   Because f decreases as you move away from $(-2.5, 4.5)$ in any direction, you would expect that f has a local maximum there.

3. Consider the function

$$f(x,y) = x^4 + y^4 - 4xy + 1.$$

(a) (5 points) Verify that the critical points of f are $(0,0)$, $(1,1)$, and $(-1,-1)$.

Note that

$$\nabla f = (4x^3 - 4y, 4y^3 - 4x).$$

Critical points satisfy $\nabla f = (0,0)$, which we can rewrite as

$$x^3 = y, \qquad y^3 = x.$$

The substituting the first equation in to the second, we get

$$x^9 = x \Leftrightarrow 0 = x^9 - x = x(x^8 - 1).$$

The only solutions to this equation are $x = 0, 1, -1$. If $x = 0$ we get $y = 0$, if $x = 1$ we get $y = 1$, and if $x = -1$ we get $y = -1$. Thus $(0,0)$, $(1,1)$ and $(-1,-1)$ are the only critical points.

(b) (5 points) Classify these critical points as local maxima, local minima, or saddle points.

First we compute second derivatives and the discriminant:

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \quad \frac{\partial^2 f}{\partial y^2} = 12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = -4, \quad D = 144x^2y^2 - 16.$$

Plugging in our critical points, we see $D(0,0) = -16 < 0$, so $(0,0)$ is a saddle point. Next, $D(1,1) = 140 > 0$ and $\partial^2 f/\partial x^2(1,1) = 12 > 0$, so $(1,1)$ is a local minimum. Finally, $D(-1,-1) = 140 > 0$ and $\partial^2 f/\partial x^2 = 12 > 0$, so $(-1,-1)$ is also a local minimum.

4. (10 points) Find the absolute maximum of $f(x,y) = xy$ on the ellipse $g(x,y) = x^2 + 4y^2 \leq 1$.

First we look for interior critical points:

$$(0,0) = \nabla f = (y,x) \Leftrightarrow x = 0 = y.$$

However, one can check that $D \equiv -1$ in this case, so $(0,0)$ is a saddle point and the maximum must occur on the boundary. Next we use Lagrange multipliers:

$$\nabla f = \lambda \nabla g, \qquad g = 1,$$

which we rewrite as

$$y = 2\lambda x, \qquad x = 8\lambda y, \qquad x^2 + 4y^2 = 1.$$

Then either $x = 0$ (which implies $y = 0$, a case we've already examined), or $16\lambda^2 = 1$. We plug $\lambda = \pm 1/4$ back into our equations to get $x = \pm 2y$, so

$$1 = x^2 + 4y^2 = 4y^2 + 4y^2 \Leftrightarrow y = \pm\frac{1}{\sqrt{8}}.$$

This in turn gives $x = \pm 1/\sqrt{2}$, and so we have four candidates for maximum points:

$$(1/\sqrt{2}, \sqrt{2}), (-1/\sqrt{2}, \sqrt{2}), (1/\sqrt{2}, -\sqrt{2}), (-1/\sqrt{2}, -\sqrt{2}).$$

Testing these, we find maximal function values (of 1/4) at $\pm(1/\sqrt{2}, 1/\sqrt{2})$. The other two points are minima.

5. (a) (5 points) Where D is the square $\{1 \leq x \leq 2, -2 \leq y \leq -1\}$, evaluate

$$\int\int_D [x^2y + yx^3] dA.$$

We'll integrate with respect to x first, but in this case it doesn't make too much difference.

$$\begin{aligned}\int\int_D [x^2y + yx^3] dA &= \int_{-2}^{-1}\int_1^2 [x^2y + yx^3] dx dy = \int_{-2}^{-1} [\frac{1}{3}x^3y\Big|_1^2 + \frac{1}{4}x^4y\Big|_1^2 dy \\ &= \int_{-2}^{-1} [8y/3 + 15y/4] dy = \frac{73}{12}\int_{-2}^{-1} y dy = \frac{73}{24}\, y^2\big|_{-2}^{-1} = -\frac{73}{8}.\end{aligned}$$

(b) (5 points) Set up, but do **not** evaluate, the integral $\int\int_D \sqrt{1+x^2+y^2}dA$, where D is the domain bounded by the curves $y = x+1$ and $x = -y^2$.

We'll integrate first with respect to x (taking horizontal slices), because it's much easier. It helps to draw the domain. It's a region bounded on the left by a slanted line and on the right by a parabola opening to the left. One can find the intersection points of these two curves by setting

$$-y^2 = x = y-1 \Leftrightarrow 0 = y^2+y-1 \Leftrightarrow y = \frac{-1\pm\sqrt{5}}{2}.$$

This gives the upper and lower limits of the y-integration. For a given y, the lower limit in the x-integral is the left-most point, which is $x = y-1$, and the upper is the right-most point, which is $x = -y^2$. Thus we have

$$\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} \int_{y-1}^{-y^2} \sqrt{1+x^2+y^2}dxdy.$$