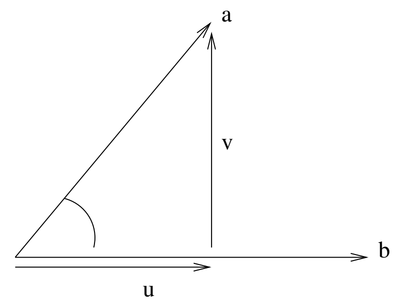
Solution to the Midterm Exam
Math 210
Feb. 16, 2005

1. Consider the vectors $\vec{a} = (2, 1, 3)$ and $\vec{b} = (-1, 0, 1)$.

   (a) (3 points) Compute the cross product $\vec{a} \times \vec{b}$.
   $\vec{a} \times \vec{b} = (2, 1, 3) \times (-1, 0, 1) = (1 - 0, -3 - 2, 0 + 1) = (1, -5, 1)$.

   (b) (3 points) Find a vector $\vec{x}$ which is perpendicular to $\vec{a}$ and verify that $\vec{x} \perp \vec{a}$. (There are many correct answers.)
   You want to find $\vec{x}$ so that $\vec{x} \cdot \vec{a} = 0$. Notice that $\vec{x} = (-1, 2, 0)$ works.

   (c) (4 points) Write $\vec{a}$ as a sum $\vec{a} = \vec{u} + \vec{v}$ where $\vec{u}$ is parallel to $\vec{b}$ and $\vec{v}$ is perpendicular to $\vec{b}$. (Hint: you only need to find on of $\vec{u}$ and $\vec{v}$. It might help to draw a picture.)
   Here's the picture you might have drawn:



   Ok, so this picture indicates

$$\begin{aligned} \vec{u} &= |\vec{a}| \cos(\theta) \frac{\vec{b}}{|\vec{b}|} = |\vec{a}| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}\vec{b}|} \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} \\ &= \frac{(2, 1, 3) \cdot (-1, 0, 1)}{2} (-1, 0, 1) = (-1/2, 0, 1/2). \end{aligned}$$

   Then

$$\vec{v} = \vec{a} - \vec{u} = (5/2, 1, 5/2).$$

2. Consider the curve $c(t)$ given by

$$c(t) = (t \cos t, t \sin t, t).$$

   (a) (3 points) Find the velocity and acceleration vectors of this curve.
   The velocity is

$$c'(t) = \frac{d}{dt}(t \cos t, t \sin t, t) = (\cos t - t \sin t, \sin t + t \cos t, 1),$$

   and the acceleration is

$$c''(t) = \frac{d}{dt}(c') = \frac{d}{dt}(\cos t - t \sin t, \sin t + t \cos t, 1) = (-2 \sin t - t \cos t, 2 \cos t - t \sin t, 0).$$

   (b) (4 points) Is the tangent line to c ever parallel to the $x - y$ plane? Be sure to explain your answer.
   In order for the tangent line to c to be parallel to the $x - y$ plane, we would need c' to be a horizontal vector. In other words, we need the third component of c' to be 0. This never happens; the third component of c' is 1. So the tangent line to c is never parallel to the $x - y$ plane.

   (c) (3 points) Set up, but do not evaluate, the integral to compute the arclength of c for $0 \leq t \leq \pi$.

$$\begin{aligned} \text{length} &= \int_0^\pi |c'(t)| dt = \int_0^\pi \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 1} dt \\ &= \int_0^\pi \sqrt{\cos^2 t - 2t \cos t \sin t + t^2 \sin^2 t + \sin^2 t + 2t \cos t \sin t + t^2 \cos^2 t + 1} dt = \int_0^\pi \sqrt{2 + t^2} dt. \end{aligned}$$

3. Consider the planes Π_1 and Π_2, given as follows. The first plane Π_1 passes through $p = (1, 2, 3)$ and has the normal vector $\vec{n} = (1, 0, -1)$. The second plane Π_2 is given by the linear equation $x + y + z = 1$.

   (a) (5 points) Explain how one can tell that Π_1 and Π_2 are not parallel, and compute the cosine of the angle θ between them.

   The normal vector to Π_1 is $\vec{n} = (1, 0, -1)$ and the normal vector to Π_2 is $\vec{m} = (1, 1, 1)$. These two vectors are not parallel, so the planes are not parallel If θ is the angle between them, then

$$\cos\theta = \frac{\vec{n} \cdot \vec{m}}{|\vec{n}||\vec{m}|} = \frac{(1, 0, -1) \cdot (1, 1, 1)}{\sqrt{2}\sqrt{3}} = 0.$$

   (b) (5 points) The two planes Π_1 and Π_2 intersect in a line l. Find a parameterization for l.

   First observe that the equation of Π_1 is

$$x - z = (1, 0, -1) \cdot (1, 2, 3) = -2.$$

   To find the line, we need a point and a direction. Let's find the point first, by looking for simultaneous solutions of

$$x - z = -2 \qquad x + y + z = 1.$$

   First we try setting $y = 0$, which gives us

$$x - z = -2 \qquad x + z = 1.$$

   Now we add the two equations, which gives us $2x = -1$, or $x = -1/2$, and so $z = 3/2$. Thus a point on the line l is $q = (-1/2, 0, 3/2)$. Now we can find a direction by taking the cross product of the two normal vectors:

$$\vec{v} = \vec{n} \times \vec{m} = (1, 0, -1) \times (1, 1, 1) = (1, -2, 1).$$

   Finally, we can parameterize the line by

$$l(t) = q + t\vec{v} = (-1/2, 0, 3/2) + t(1, -2, 1).$$

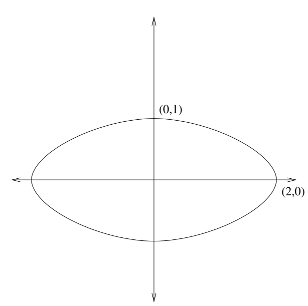
4. Consider the function $f(x, y) = x^2 + 4y^2$.

   (a) (5 points) Sketch the $f = 4$ level set.

   We can rewrite the $f = 4$ level set as

$$1 = \frac{1}{4}(x^2 + 4y^2) = \frac{x^2}{4} + y^2,$$

   which one can recognize as an ellipse with horizontal axis 2 and vertical axis 1. Here is a sketch.



   (b) (5 points) For which values of z does the level set $f = z$ not contain any points? Be sure to explain your answer.

   We want to find values of z such that the equation

$$x^2 + 4y^2 = z$$

   has no solutions. The left hand side can be zero, or any positive number, but it can't be negative. So the level sets $f = z$ for any negative z do not contain any points, while all other level sets contain at least one point.

5. Consider the function $f(x, y) = x^2y + y^2x$.

(a) (5 points) Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y + y^2x) = 2xy + y^2 \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y + y^2x) = x^2 + 2xy$$

(b) (5 points) Find the points (x, y) where the tangent plane to the graph of f is parallel to the $x - y$ plane. Be sure to explain your answer.

We know that the two tangent directions of the graph of f are

$$(1, 0, \frac{\partial f}{\partial x}) = (1, 0, 2xy + y^2)$$

and

$$(0, 1, \frac{\partial f}{\partial y}) = (0, 1, x^2 + 2xy)$$

so the normal is given by their cross product, which is

$$\vec{n} = (1, 0, 2xy + y^2) \times (0, 1, x^2 + 2xy) = (-2xy - y^2, -x^2 - 2xy, 1).$$

So the tangent plane to the graph is f is horizontal precisely when the partial derivatives both vanish:

$$2xy + y^2 = 0 = x^2 + 2xy.$$

We can rearrange this to read

$$-y^2 = 2xy = -x^2 \Leftrightarrow x = 0 = y.$$

So the only solution is $x = 0, y = 0$.