Practice Problems

Math 210

April 22, 2005

These problems are not in any particular order. The exam will be shorter.

1. For each of the following pairs of vectors, compute $\vec{u} \cdot \vec{v}$, $\vec{u} \times \vec{v}$ and the orthogonal projection of $\vec{u}$ onto $\vec{v}$.
   (a) $\vec{u} = (1, 0, 1), \vec{v} = (1, 1, 0)$
   (b) $\vec{u} = (0, 1, 0), \vec{v} = (0, 1, -1)$

2. Let Π_1 be the plane through $(1, 1, 1)$ with normal vector $\vec{n_1} = (1, -1, 0)$, and let Π_2 be the plane defined by $x + y + z = 1$.
   (a) Write down a linear equation for Π_1.
   (b) Find the cosine angle between Π_1 and Π_2.
   (c) Parameterize the line l of intersection between Π_1 and Π_2.
   (d) Find the distance between Π_2 and $(1, 1, 1)$.

3. Consider the curve $c(t) = (\cos(2t), \sin(t))$ for $0 \le t \le 2\pi$.
   (a) Sketch this curve.
   (b) Write down the tangent line to c at the point $(-1, 1)$. (This point corresponds to the parameter value $t = \pi/2$.
   (c) Is the tangent line to this curve ever parallel to the line $y = -x$? Be sure to explain your answer.
   (d) Set up, but do not evaluate, the integral to compute the arclenth of this curve.

4. Consider the function
$$f(x, y) = \int_y^{e^x} \sqrt{1 + t^2} dt.$$
   (a) Compute the partial derivatives of f.
   (b) Does f have any critical points? Be sure to explain your answer.

5. Consider the function $f(x, y) = x^2 y - xy^3$.
   (a) Does f have an upper or lower bound? Explain your answer.
   (b) Compute the partial derivatives of f.
   (c) Compute the directional derivative of f in the $(1/\sqrt{2}, -1/\sqrt{2})$ direction, at the point $(1, 1)$.
   (d) Find the direction of steepest ascent for f, starting at $(1, 1)$. Make sure to write down a unit vector.
   (e) Notice that $f(1, 1) = 0$. Write down the equation of the tangent line to the $f = 0$ level set, at the point $(1, 1)$.

6. Consider the function
$$f(x, y) = e^{xy^3 - x^2}.$$
   (a) Find and classify all the critical points of f.
   (b) Find the absolute minimum of f restricted to the square $0 \le x \le 1$, $0 \le y \le 1$.

7. Recall that two tangent directions to the graph of f are given by the vectors
$$(1, 0, \frac{\partial f}{\partial x}), \qquad (0, 1, \frac{\partial f}{\partial y}).$$
   (a) Compute a normal vector for the graph.
   (b) Can the tangent plane of the graph ever be parallel to the $x - z$ plane? Explain your answer.

8. Find the absolute maximum and minimum of the function $f = xy$ on the ellipse $4x^2 + y^2 = 4$.

9. Evaluate $\int_D \sqrt{4 - x^2 - y^2} dA$ where $D = \{(x, y) \mid 1 \le x^2 + y^2 \le 4, y > 0$.

10. Set up, but do not evaluate $\int_D xe^{x^2y}dA$, where D is the region bounded by the curves $y = x^3$ and $y = x$. (Be careful of signs.)

11. Evaluate $\int\int_D \sqrt{1-x^2}dA$ where D is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

12. Consider the domain $D := \{(x,y) \mid |x+y| \leq 1\}$.

    (a) Sketch D.

    (b) If we change variables by $u = \frac{1}{\sqrt{2}}(x+y), v = \frac{1}{\sqrt{2}}(-x+y)$, what is D in the u, v coordinate system?

    (c) Set up, but do not evaluate, the integral $\int\int_D \cos(\pi x - \pi y)dA$ in the (u,v) coordiante system.

13. Consider the vector field $\vec{F} = (-y, x)$.

    (a) Sketch $\vec{F}$.

    (b) Compute $\int_\gamma \vec{F} \cdot d\vec{s}$ where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$.

14. Consider the vector field $\vec{F} = (2xe^{x^2+y^2}, 2ye^{x^2+y^2} + \cos y)$.

    (a) Show that $\vec{F} = \nabla f$ for some f.

    (b) Find a function f such that $\vec{F} = \nabla f$.

    (c) Compute $\int_\gamma \vec{F} \cdot d\vec{s}$, where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi/2$. (Hint: you don't need to actually do an integral.)

15. Consider $\vec{F} = (-y + x^2 - y\cos(xy), x - y^3 - x\cos(xy))$.

    (a) Is $\vec{F} = \nabla f$ for some f? Be sure to explain your answer.

    (b) Compute $\int_\gamma \vec{F} \cdot d\vec{s}$. (Hint: don't actually compute the line integral.)

16. Consider the vector field $\vec{F} = (z + x^3 - yze^{xyz}, y - xze^{xyz}, -x + z^2 - xye^{xyyz})$.

    (a) Compute $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$.

    (b) Is $\vec{F} = \nabla f$ for some f? Be sure to explain your answer.

    (c) Compute $\int\int_\Sigma \nabla \times \vec{F} \cdot \vec{n}dA$, where Σ is the upper unit hemisphere, centered at $(0,0,0)$, with the outward unit normal.

17. Consider the surface $\vec{r}(u,v) = (u\cos(v), u\sin(v), v)$.

    (a) Compute the tangent vectors $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$.

    (b) Verify that this is a good parameterization, by checking that $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$ are never parallel.

    (c) Find the equation of the tangent plane to $\vec{r}$ for the parameter values $u = 1, v = \pi$.

    (d) Is the tangent plane ever parallel to the $x - y$ plane? Be sure to explain your answer.

    (e) What is this surface? Can you draw a sketch of it? (Hint: fix a value of u, for instance $u = 1$ or $u = 0$, and draw the resulting curve.)

18. Compute $\int\int_\Sigma \nabla \times \vec{F} \cdot \vec{n}dA$, where $\vec{F} = (-y, x, 0)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal.

19. Compute $\int\int_\Sigma \vec{F} \cdot \vec{n}dA$, where $\vec{F} = (x + yz - \cos y, y - e^{xz} + z^2, z - x\cos(x^2y))$ and Σ is the unit sphere (centered at the origin) with the outward unit normal.

20. Compute $\int\int_\Sigma \vec{F} \cdot \vec{n}dA$, where $\vec{F} = (x + ze^{y^2+z}, y - z\cos(x+z^2), z)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal. (Hint: what is $\vec{F}$ restricted to the plane $z = 0$?)