

Practice Problems

Math 210

April 28, 2005

1. For each of the following pairs of vectors, compute $\vec{u} \cdot \vec{v}$, $\vec{u} \times \vec{v}$ and the orthogonal projection of \vec{u} onto \vec{v} .

(a) $\vec{u} = (1, 0, 1), \vec{v} = (1, 1, 0)$

$$\vec{u} \cdot \vec{v} = (1, 0, 1) \cdot (1, 1, 0) = 1 + 0 + 0 = 1$$

$$\vec{u} \times \vec{v} = (1, 0, 1) \times (1, 1, 0) = (0 - 1, 1 - 0, 1 - 0) = (-1, 1, 1)$$

We'll call the projection $\text{Proj}_{\vec{v}}(\vec{u})$. To compute it, we want to compute $|\vec{u}| \cos \theta$ times a unit vector in the same direction as \vec{v} , where θ is the angle between \vec{u} and \vec{v} . We have

$$\text{Proj}_{\vec{v}}(\vec{u}) = |\vec{u}| \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{1}{2} \vec{v} = (1/2, 1/2, 0).$$

(b) $\vec{u} = (0, 1, 0), \vec{v} = (0, 1, -1)$

$$\vec{u} \cdot \vec{v} = 0 + 1 + 0 = 1$$

$$\vec{u} \times \vec{v} = (0, 1, 0) \times (0, 1, -1) = (-1 - 0, 0 - 0, 0 - 0) = (-1, 0, 0)$$

$$\text{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{1}{2} \vec{v} = (0, 1/2, -1/2)$$

2. Let Π_1 be the plane through $(1, 1, 1)$ with normal vector $\vec{n}_1 = (1, -1, 0)$, and let Π_2 be the plane defined by $x + y + z = 1$.

- (a) Write down a linear equation for Π_1 .

Denote the normal to Π_1 as $\vec{n}_1 = (1, -1, 0)$. Then (x, y, z) is in Π_1 if and only if

$$((x, y, z) - (1, 1, 1)) \cdot \vec{n}_1 = 0 \Leftrightarrow x = y.$$

- (b) Find the cosine angle between Π_1 and Π_2 .

The angle θ between Π_1 and Π_2 is the same as the angle between \vec{n}_1 and \vec{n}_2 , where $\vec{n}_2 = (1, 1, 1)$ is the normal to Π_2 . Thus we have

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = 0 \Leftrightarrow \theta = 90^\circ.$$

- (c) Parameterize the line l of intersection between Π_1 and Π_2 .

The direction vector of this line is

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = (1, -1, 0) \times (1, 1, 1) = (-1, -1, 2).$$

Next we find a common point, which amounts to finding a simultaneous solution of the following two equations:

$$x = y, \quad x + y + z = 1.$$

One can check that $(1/2, 1/2, 0)$ works. Then the line l is given by

$$l(t) = (1/2, 1/2, 0) + t(-1, -1, 2).$$

- (d) Find the distance between Π_2 and $(1, 1, 1)$.

We first pick a base point $(1, 0, 0)$ is Π_2 (it doesn't matter which basepoint we pick). Then a displacement vector from $(1, 1, 1)$ to Π_2 is $(1, 1, 1) - (1, 0, 0) = (0, 1, 1)$, and we can compute the distance as an orthogonal projection of this displacement vector:

$$\text{dist} = |\text{Proj}_{\vec{n}_2}(0, 1, 1)| = \frac{|(0, 1, 1) \cdot \vec{n}_2|}{|\vec{n}_2|} = \frac{2}{\sqrt{3}}.$$

3. Consider the curve $c(t) = (\cos(2t), \sin(t))$ for $0 \leq t \leq 2\pi$.

(a) Sketch this curve.

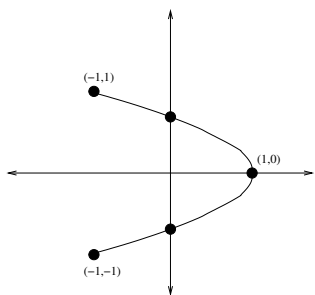


Figure 1: figure for problem 3

The marked points in this sketch correspond are $(1, 0)$ (corresponding to $t = 0, \pi$), $(-1, 1)$ (corresponding to $t = \pi/2$, where the curve turns around), $(-1, -1)$ (corresponding to $t = 3\pi/2$, where the curve turns around again), $(0, 1/\sqrt{2})$ (corresponding to $t = \pi/4, 3\pi/4$), and $(0, -1/\sqrt{2})$ (corresponding to $t = 5\pi/4, 7\pi/4$).

(b) Write down the tangent line to c at the point $(-1, 1)$. (This point corresponds to the parameter value $t = \pi/2$.)

We can see from the sketch that c doesn't really have a tangent line at that point, because it turns around there. We can also see this by looking at c' . First we compute the velocity vector: $c'(t) = (-2\sin(2t), \cos(t))$. Evaluating this at $t = \pi/2$, we get a direction vector for the tangent line $\vec{v} = (-2\sin(\pi), \cos(\pi/2)) = (0, 0)$. This means there is no tangent line.

(c) Is the tangent line to this curve ever parallel to the line $y = -x$? Be sure to explain your answer.

The tangent line is parallel to $y = -x$ precisely when $c'(t)$ is parallel to $(1, -1)$, which is a vector parallel to $y = -x$. This happens when $\cos t = 2\sin(2t) = 4\cos t \sin t$. There are two possibilities: either $\cos t = 0$, where c doesn't really have a tangent line, or $\sin t = 1/4$. The latter equation has two solutions, $t \approx .2527, 2.889$, so the tangent line to c is parallel to $y = -x$ for those values of t .

(d) Set up, but do not evaluate, the integral to compute the arclength of this curve.

$$\text{Length}(c) = \int_0^{2\pi} |c'(t)| dt = \int_0^{2\pi} \sqrt{4\sin^2(2t) + \cos^2 t} dt$$

4. Consider the function

$$f(x, y) = \int_y^{e^x} \sqrt{1+t^2} dt.$$

(a) Compute the partial derivatives of f .

We use the fundamental theorem of calculus:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_y^{e^x} \sqrt{1+t^2} dt = \sqrt{1+e^{2x}}(e^x)' = e^x \sqrt{1+e^{2x}}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_y^{e^x} \sqrt{1+t^2} dt = -\sqrt{1+y^2},$$

(b) Does f have any critical points? Be sure to explain your answer.

Notice that $\partial f/\partial x > 0$ and $\partial f/\partial y < 0$, so there are no points satisfying $(\partial f/\partial x, \partial f/\partial y) = (0, 0)$. Thus there are no critical points.

5. Consider the function $f(x, y) = x^2y - xy^3$.

(a) Does f have an upper or lower bound? Explain your answer.

Notice that if we plug in $y = x$, we get $x^3 - x^4 \rightarrow -\infty$ as $|x| \rightarrow \infty$. Also, if we plug in $y = -x$ we get $x^4 - x^3 \rightarrow \infty$ as $|x| \rightarrow \infty$. Thus f has neither a lower bound nor an upper bound.

(b) Compute the partial derivatives of f .

$$\frac{\partial f}{\partial x} = 2xy - y^3, \quad \frac{\partial f}{\partial y} = x^2 - 3xy^2$$

(c) Compute the directional derivative of f in the $(1/\sqrt{2}, -1/\sqrt{2})$ direction, at the point $(1, 1)$.
First evaluate the partial derivatives at $(1, 1)$:

$$\frac{\partial f}{\partial x}(1, 1) = 2 - 1 = 1, \quad \frac{\partial f}{\partial y}(1, 1) = 1 - 3 = -2.$$

Then the directional derivative is

$$\nabla_{(1/\sqrt{2}, -1/\sqrt{2})} f(1, 1) = (1, -2) \cdot (1/\sqrt{2}, -1/\sqrt{2}) = 3/\sqrt{2}.$$

(d) Find the direction of steepest ascent for f , starting at $(1, 1)$. Make sure to write down a unit vector.

The gradient $\nabla f = (\partial f/\partial x, \partial f/\partial y)$ always points in the direction of steepest ascent. The unit vector in this direction is

$$\frac{\nabla f(1, 1)}{|\nabla f(1, 1)|} = (1/\sqrt{5}, -2/\sqrt{5}).$$

(e) Notice that $f(1, 1) = 0$. Write down the equation of the tangent line to the $f = 0$ level set, at the point $(1, 1)$.

The level sets are always perpendicular to ∇f . So we choose a vector perpendicular to $\nabla f(1, 1) = (1, -2)$, namely $\vec{v} = (2, 1)$. Then the tangent line is parameterized as

$$l(t) = (1, 1) + t(2, 1),$$

or equivalently as

$$y - 1 = \frac{1}{2}(x - 1).$$

6. Consider the function

$$f(x, y) = e^{xy^3 - x^2}.$$

(a) Find and classify all the critical points of f .

First we take the partial derivatives:

$$\frac{\partial f}{\partial x} = (y^3 - 2x)e^{xy^3 - x^2}, \quad \frac{\partial f}{\partial y} = 3xy^2 e^{xy^3 - x^2}.$$

The critical points occur when $\nabla f = (\partial f/\partial x, \partial f/\partial y) = (0, 0)$. Since the exponential is never zero, we require $y^3 - 2x = 0$ and $3xy^2 = 0$. The second equation implies either $x = 0$ or $y = 0$. The first equation implies that either x or y is zero, the so is the other one. Thus the only critical point is $(0, 0)$.

Next we have to see what kind of critical point it is. We will need the second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2}(0, 0) = (2 + (y^3 - 2x)^2)e^{xy^3 - x^2} \Big|_{(0,0)} = 2, \quad \frac{\partial^2 f}{\partial y^2}(0, 0) = (6xy + 9x^2y^4)e^{xy^3 - x^2} \Big|_{(0,0)} = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = (3y^2 + 3xy^2(y^3 - 2x))e^{xy^3 - x^2} \Big|_{(0,0)} = 0.$$

Thus the discriminant is $D = (\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2) - (\partial^2 f/\partial x \partial y)^2 = 0$, so the second derivative test is inconclusive. Looking a little closer at the function, we see that (to second order), the function looks like $f \approx x^2$ near $(0, 0)$, so the origin is neither a maximum nor a minimum.

(b) Find the absolute minimum of f restricted to the square $0 \leq x \leq 1, 0 \leq y \leq 1$.

Notice that the exponential function is monotone increasing, so it will suffice to find minima of $g(x, y) = xy^3 - x^2$. At the only critical point $(0, 0)$, we have $g = 0$ so $f = 1$. Also notice that if we restrict to $x = 0$, we get $g = 0$, and so $f = 1$ along this edge as well.

Next we restrict to $x = 1$, where $g = y^3 - 1$. This has a critical point at $y = 0$, with value $g = -1$ (and so $f = 1/e$). We also have $g(1, 1) = 0$, and so $f(1, 1) = 1$. If we restrict to $y = 0$, then we get $g = -x^2$, which has a minimum value of $g = -1$ (and so $f = 1/e$) at $(1, 0)$. Finally, we restrict to $y = 1$, where $g = x - x^2$. This has a critical point at $x = 1/2$, where $g(1/2, 1) = 1/4$ (and $f(1/2, 1) = e^{1/4}$). Thus our candidates for minimal values of f are $f = 1$ (along the $x = 0$ and edge and at $(1, 1)$), $f = 1/e$ (at $(1, 0)$) and $f = e^{1/4}$ (at $(1/2, 1)$). The smallest of these candidates is $f = 1/e$, which occurs at $(1, 0)$.

7. Recall that two tangent directions to the graph of f are given by the vectors

$$\left(1, 0, \frac{\partial f}{\partial x}\right), \quad \left(0, 1, \frac{\partial f}{\partial y}\right).$$

(a) Compute a normal vector for the graph.

The normal is given by

$$\left(1, 0, \frac{\partial f}{\partial x}\right) \times \left(0, 1, \frac{\partial f}{\partial y}\right) = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right).$$

(b) Can the tangent plane of the graph ever be parallel to the $x - z$ plane? Explain your answer.

The normal to $x - z = 0$ is $(1, 0, 1)$, so we'd want

$$\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = \lambda(1, 0, 1) \Leftrightarrow \frac{\partial f}{\partial x} = -1, \frac{\partial f}{\partial y} = 0.$$

This is possible; in fact we can choose $f = \cos x$ and $(x, y) = (\pi/2, 0)$.

8. Find the absolute maximum and minimum of the function $f = xy$ on the ellipse $4x^2 + y^2 = 4$.

We use Lagrange multipliers. Letting $g = 4x^2 + y^2$, we see that the ellipse is the level set $g = 4$. Critical points of f on the ellipse then satisfy

$$\nabla f = \lambda \nabla g, \quad g = 4,$$

where λ is some nonzero constant. We rewrite this system as

$$(y, x) = \lambda(8x, 2y), \quad 4x^2 + y^2 = 4.$$

The first equations combine to give us $x = 16\lambda^2 x$. If $x \neq 0$, this yields $\lambda = \pm 1/4$, which we can then plug into the last equation (using $y = \pm 2x$):

$$4 = 4x^2 + y^2 = 4x^2 + (\pm 2x)^2 = 8x^2 \Leftrightarrow x = \pm \frac{1}{\sqrt{2}}.$$

This gives four critical points $\pm(1/\sqrt{2}, 1/\sqrt{2})$ and $\pm(-1/\sqrt{2}, 1/\sqrt{2})$. We still have to consider the case of $x = 0$, but in this case we also have $y = 0$, which doesn't satisfy the constraint. One can check that f achieves its maximum value of $1/2$ at $\pm(1/\sqrt{2}, 1/\sqrt{2})$, while it achieves its minimum value of $-1/2$ at $\pm(-1/\sqrt{2}, 1/\sqrt{2})$.

9. Evaluate $\int_D \sqrt{4 - x^2 - y^2} dA$ where $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y > 0\}$.

We will use polar coordinates. In these coordinates, D is the region $1 \leq r \leq 2$ and $f = \sqrt{4 - r^2}$. Then

$$\iint_D f dA = \int_0^{2\pi} \int_1^2 r \sqrt{4 - r^2} dr d\theta = -\frac{1}{2} \int_0^{2\pi} \int_3^0 \sqrt{u} du d\theta = 2\pi\sqrt{3}.$$

10. Set up, but do not evaluate $\int_D x e^{x^2 y} dA$, where D is the region bounded by the curves $y = x^3$ and $y = x$. (Be careful of signs.)

We will use vertical slices (integrating with respect to y first). First notice that the curves intersect at $(0, 0)$, $(1, 1)$, and $(-1, -1)$. The region D splits into two parts: the part with $-1x < 0$, where $x < x^3$ and the part with $0 < x < 1$, where $x > x^3$. Thus we have

$$\iint_D x e^{x^2 y} dA = \int_{-1}^0 \int_x^{x^3} x e^{x^2 y} dy dx + \int_0^1 \int_{x^3}^x x e^{x^2 y} dy dx.$$

11. Evaluate $\int \int_D \sqrt{1 - x^2} dA$ where D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

The triangle D is bounded by the axes and the line $x + y = 1$. We will evaluate this integral using vertical slices (integrating with respect to y first). Then

$$\begin{aligned} \iint_D \sqrt{1 - x^2} dA &= \int_0^1 \int_0^{1-x} \sqrt{1 - x^2} dy dx = \int_0^1 (1 - x) \sqrt{1 - x^2} dx = \int_0^1 \sqrt{1 - x^2} dx - \int_0^1 x \sqrt{1 - x^2} dx \\ &= \int_0^1 \sqrt{1 - x^2} dx - \frac{1}{2} \int_0^1 \sqrt{u} du = \int_0^1 \sqrt{1 - x^2} dx - \frac{1}{3} u^{3/2} \Big|_0^1 = \int_0^1 \sqrt{1 - x^2} dx - \frac{1}{3} \\ &= \int_0^{\pi/2} \sqrt{1 - \sin^2 \theta} \cos \theta d\theta - \frac{1}{3} = \int_0^{\pi/2} \cos^2 \theta d\theta - \frac{1}{3} = \frac{1}{2} \int_0^{\pi/2} (1 + 2 \cos(2\theta)) d\theta - \frac{1}{3} \\ &= \frac{\pi}{4} + \frac{1}{2} \sin(2\theta) \Big|_0^{\pi/2} - \frac{1}{3} = \frac{\pi}{4} - \frac{1}{3} \end{aligned}$$

12. Consider the domain $D := \{(x, y) \mid |x + y| \leq 1\}$.

(a) Sketch D .

The region D is the strip between the two lines drawn below.

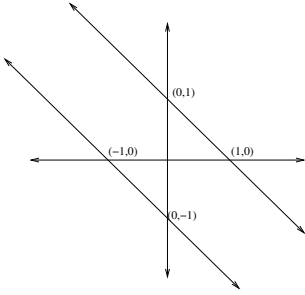


Figure 2: figure for problem 12

(b) If we change variables by $u = \frac{1}{\sqrt{2}}(x + y)$, $v = \frac{1}{\sqrt{2}}(-x + y)$, what is D in the u, v coordinate system?

Notice that $x + y = \sqrt{2}u$, so D is the domain $|u| \leq 1/\sqrt{2}$.

(c) Set up, but do not evaluate, the integral $\iint_D \cos(\pi x - \pi y) dA$ in the (u, v) coordinate system.

Observe that

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{2}} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{1}{\sqrt{2}}.$$

So the Jacobian for the change of variables is

$$J = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = \frac{1}{2} + \frac{1}{2} = 1.$$

Also, in these variables, $\cos(\pi x - \pi y) = \cos(-(\pi/\sqrt{2})v) = \cos((\pi/\sqrt{2})v)$. Thus the integral is

$$\iint_D \cos(\pi x - \pi y) dA = \int_{-\infty}^{\infty} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \cos((\pi/\sqrt{2})v) dudv.$$

13. Consider the vector field $\vec{F} = (-y, x)$.

(a) Sketch \vec{F} .

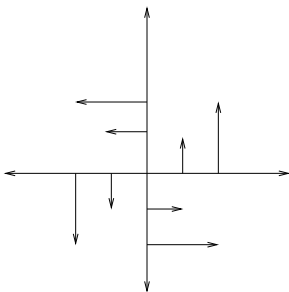


Figure 3: figure for problem 13

(b) Compute $\int_{\gamma} \vec{F} \cdot d\vec{s}$ where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$.

Observe that $\gamma' = (-\sin t, \cos t)$. So we have

$$\int_{\gamma} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\gamma(t)) \cdot \gamma'(t) dt = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt = \int_0^{2\pi} dt = 2\pi.$$

14. Consider the vector field $\vec{F} = (2xe^{x^2+y^2}, 2ye^{x^2+y^2} + \cos y)$.

- (a) Show that $\vec{F} = \nabla f$ for some f .

We have to check that $\partial F_2/\partial x = \partial F_1/\partial y$. We can compute:

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(2ye^{x^2+y^2} + \cos y) = 4xye^{x^2+y^2}, \quad \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(2xe^{x^2+y^2}) = 4xye^{x^2+y^2}.$$

- (b) Find a function f such that $\vec{F} = \nabla f$.

We have to find an f such that

$$\frac{\partial f}{\partial x} = 2xe^{x^2+y^2}, \quad \frac{\partial f}{\partial y} = 2ye^{x^2+y^2} + \cos y.$$

Integrating the first equation with respect to x , we get $f = e^{x^2+y^2} + g(y)$ for some unknown function $g(y)$. The second equation then says $g'(y) = \cos y$, which we can integrate to get $g = \sin y$. Thus we have $f(x, y) = e^{x^2+y^2} + \sin y$ (up to adding a constant).

- (c) Compute $\int_{\gamma} \vec{F} \cdot d\vec{s}$, where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi/2$. (Hint: you don't need to actually do an integral.)

The endpoints of γ are $\gamma(\pi/2) = (0, 1)$ and $\gamma(0) = (1, 0)$. Because $\vec{F} = \nabla f = \nabla(e^{x^2+y^2} + \sin y)$, we can use the fundamental theorem of calculus:

$$\int_{\gamma} \vec{F} \cdot d\vec{s} = f(0, 1) - f(1, 0) = (e^1 + \sin(1)) - (e^1 + \sin(0)) = \sin(1).$$

15. Consider $\vec{F} = (-y + x^2 - y \cos(xy), x - y^3 - x \cos(xy))$.

- (a) Is $\vec{F} = \nabla f$ for some f ? Be sure to explain your answer.

No. Indeed, $\partial F_2/\partial x = 1$, while $\partial F_1/\partial y = -1$. These two would have to be equal if $\vec{F} = \nabla f$.

- (b) Compute $\int_{\gamma} \vec{F} \cdot d\vec{s}$. (Hint: don't actually compute the line integral.)

Forgot to say that γ is supposed to be the unit circle (centered at the origin, but that part isn't important). Then $\gamma = \partial D$, where D is the unit disc, so we can use Green's theorem:

$$\int_{\gamma} \vec{F} \cdot d\vec{s} = \int \int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = 2 \int \int_D dA = 2\pi.$$

16. Consider the vector field $\vec{F} = (z + x^3 - yze^{xyz}, y - xze^{xyz}, -x + z^2 - xye^{xyz})$.

- (a) Compute $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$.

First we compute the curl:

$$\nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = (0, 2, 0).$$

Next we compute the divergence:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + 2z + 1 - (x^2y^2 + y^2z^2 + x^2z^2)e^{xyz}.$$

- (b) Is $\vec{F} = \nabla f$ for some f ? Be sure to explain your answer.

No. If $\vec{F} = \nabla f$, then $\nabla \times \vec{F} = 0$. However, we just computed that the curl of \vec{F} is not 0.

- (c) Compute $\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA$, where Σ is the upper unit hemisphere, centered at $(0, 0, 0)$, with the outward unit normal.

Let D be the unit disc in the $x - y$ plane (with the upward normal $\vec{n} = (0, 0, 1)$), and observe that Σ and D have the same boundary curve γ , which is the unit circle in the $x - y$ plane. Then by Stokes theorem

$$\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA = \int_{\gamma} \vec{F} \cdot d\vec{s} = \int \int_D \nabla \times \vec{F} \cdot \vec{n} dA = \int \int_D (0, 2, 0) \cdot (0, 0, 1) dA = 0.$$

17. Consider the surface $\vec{r}(u, v) = (u \cos(v), u \sin(v), v)$.

- (a) Compute the tangent vectors $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$.

$$\frac{\partial\vec{r}}{\partial u} = (\cos v, \sin v, 0), \quad \frac{\partial\vec{r}}{\partial v} = (-u \sin v, u \cos v, 1).$$

- (b) Verify that this is a good parameterization, by checking that $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$ are never parallel. We compute the cross product $(\partial\vec{r}/\partial u) \times (\partial\vec{r}/\partial v)$, and check that we don't get zero:

$$\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v} = (\cos v, \sin v, 0) \times (-u \sin v, u \cos v, 1) = (\sin v, -\cos v, u).$$

Notice that $\sin v$ and $\cos v$ are never simultaneously zero, so either the first or the second component of this vector is nonzero.

- (c) Find the equation of the tangent plane to \vec{r} for the parameter values $u = 1, v = \pi$.

First evaluate the normal vector we computed above for the parameter values $u = 1, v = \pi$. We get

$$\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v} = (0, 1, 1).$$

Next we evaluate $\vec{r}(1, \pi) = (-1, 0, \pi)$. The equation of the tangent plane is then

$$0 = (x - (-1), y - 0, z - \pi) \cdot (0, 1, 1) \Leftrightarrow y + z = \pi.$$

- (d) Is the tangent plane ever parallel to the $x - y$ plane? Be sure to explain your answer.

The tangent plane to the surface is parallel to the $x - y$ plane precisely when the normal is parallel to $(0, 0, 1)$. Thus we would need

$$(\sin v, -\cos v, u) = \lambda(0, 0, 1)$$

for some $\lambda \neq 0$. However, there is no value of v such that $\sin v = 0$ and $\cos v = 0$, so this never happens, and the tangent plane is never parallel to the $x - y$ plane.

- (e) What is this surface? Can you draw a sketch of it? (Hint: fix a value of u , for instance $u = 1$ or $u = 0$, and draw the resulting curve.)

This is a helicoid. If you fix a value of u , the curve you get is a helix, twisting around the z axis. (Except $u = 0$; then you just get the z axis.) The easiest way to visualize this surface is to start with the line $y = z = 0$, and raise it at a constant rate, while you twist it in the counter-clockwise direction around the z axis at the same rate. This sweeps out the surface.

18. Compute $\int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA$, where $\vec{F} = (-y, x, 0)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal.

There are a couple of ways to do this. We'll use Stokes theorem and parameterize $\partial\Sigma$ as $\gamma(t) = (\cos t, \sin t, 0)$ for $0 \leq t \leq 2\pi$. Observe that $\gamma'(t) = (-\sin t, \cos t, 0)$. Then

$$\int \int_{\Sigma} \nabla \times \vec{F} \cdot \vec{n} dA = \int_{\gamma} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0) dt = \int_0^{2\pi} dt = 2\pi.$$

19. Compute $\int_{\Sigma} \vec{F} \cdot \vec{n} dA$, where $\vec{F} = (x + yz - \cos y, y - e^{xz} + z^2, z - x \cos(x^2 y))$ and Σ is the unit sphere (centered at the origin) with the outward unit normal.

Here we use the divergence theorem. Notice that $\Sigma = \partial B$, where B is the unit ball centered at the origin. Also, $\nabla \cdot \vec{F} = 3$. Then

$$\int \int_{\Sigma} \vec{F} \cdot \vec{n} dA = \int \int_{\partial B} \vec{F} \cdot \vec{n} dA = \int \int \int_B \nabla \cdot \vec{F} dV = 3 \int \int \int_B dV = 4\pi.$$

20. Compute $\int_{\Sigma} \vec{F} \cdot \vec{n} dA$, where $\vec{F} = (x + ze^{y^2+z}, y - z \cos(x+z^2), z)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal. (Hint: what is \vec{F} restricted to the plane $z = 0$?)

Notice that $\nabla \cdot \vec{F} = 3$. We'd like to use the divergence theorem, but Σ is not a closed surface, so it's not the boundary of any solid region. However, we can add something to Σ and make the resulting union of surfaces into the boundary of a solid. In this case, we add the unit disc $D = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$, with the **downward** normal. Then we let $B_+ = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, z \geq 0\}$ be the upper half-ball. Observe that $\partial B_+ = \Sigma + D$. Then by the divergence theorem, we have

$$\int \int_{\Sigma} \vec{F} \cdot \vec{n} dA + \int \int_D \vec{F} \cdot \vec{n} dA = \int \int \int_{B_+} \nabla \cdot \vec{F} dV = \int \int \int_{B_+} 3 dV = 2\pi.$$

All that remains is to compute $\int \int_D \vec{F} \cdot \vec{n} dA$. We have

$$\int \int_D \vec{F} \cdot \vec{n} dA = \int_0^{2\pi} \int_0^1 (r \cos \theta, r \sin \theta, 0) \cdot (0, 0, -1) r dr d\theta = 0.$$

Putting everything together, we see that

$$\int \int_{\Sigma} \vec{F} \cdot \vec{n} dA = \int \int \int_{B_+} \nabla \cdot \vec{F} dV - \int \int_D \vec{F} \cdot \vec{n} dA = 2\pi.$$