

Solutions to the Practice Problems

Math 210

March 17, 2005

1. Consider the function $f(x, y) = x^3y - yx^2 + 3xy$.

(a) Write down the equation of the tangent plane to the graph of f at $(0, 1)$.

Note $f(0, 1) = 0$. Also,

$$\frac{\partial f}{\partial x} = 3x^2y - 2xy + 3y, \quad \frac{\partial f}{\partial x}(0, 1) = 3, \quad \frac{\partial f}{\partial y} = x^3 - x + 3x, \quad \frac{\partial f}{\partial y}(0, 1) = 0.$$

The tangent plane at $(0, 1)$ is given by

$$0 = (x - 0, y - 1, z - 0) \cdot \left(-\frac{\partial f}{\partial x}(0, 1), -\frac{\partial f}{\partial y}(0, 1), 1\right) = -3x + z,$$

which we can rearrange to read $z = 3x$.

(b) At which points (x, y) is the tangent plane horizontal?

We want to find points where the normal vector $(-\partial f/\partial x, -\partial f/\partial y, 1)$ is vertical, so we want $\partial f/\partial x = 0 = \partial f/\partial y$. Let's look at $\partial f/\partial y$ first:

$$0 = \frac{\partial f}{\partial y} = x^3 - x^2 + 3x = x(x^2 - x + 3) \Leftrightarrow x = 0.$$

Next we plug $x = 0$ into $\partial f/\partial x$ to see

$$0 = \frac{\partial f}{\partial x}(0, y) = 3y \Leftrightarrow y = 0.$$

Thus the only point with a horizontal tangent plane is $(0, 0)$.

2. Consider

$$f(x, y) = \int_x^y e^{t^2} dt.$$

(a) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

We use the Fundamental Theorem of Calculus:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_x^y e^{t^2} dt = -e^{x^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_x^y e^{t^2} dt = e^{y^2}.$$

(b) Does f have any critical points? Explain your answer.

Critical points satisfy $\partial f/\partial x = 0 = \partial f/\partial y$. However, both partial derivatives are exponentials, which are never zero. So there are no critical points.

3. Let some of the level sets of the function f be given by the figure below.

(a) Estimate $\frac{\partial f}{\partial x}(2, 0)$

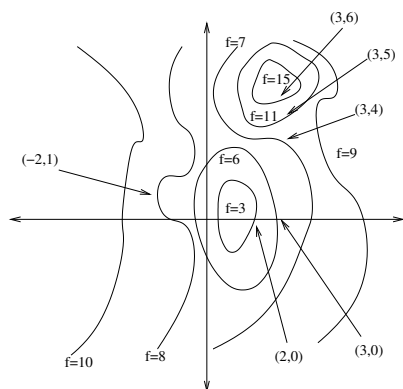
Notice that $(2, 0)$ is on the $f = 3$ level set, while $(3, 0)$ is on the $f = 6$ level set. So

$$\frac{\partial f}{\partial x}(2, 0) \approx \frac{6 - 3}{3 - 2} = 3.$$

(b) Estimate $\frac{\partial f}{\partial y}(3, 4)$

Notice that $(3, 4)$ is on the $f = 7$ level set, while $(3, 5)$ is on the $f = 11$ level set. So

$$\frac{\partial f}{\partial y}(3, 4) \approx \frac{11 - 7}{5 - 4} = 4.$$



(c) In which direction does $\nabla f(-2, 1)$ point?

Notice that $(-2, 1)$ is on the $f = 8$ level set, and this curve is vertical at $(-2, 1)$. Also, the $f = 10$ level set is to the left and the $f = 6$ and $f = 7$ level sets are to the right. Thus, because ∇f is perpendicular to level sets and points in the direction f increases, we know $\nabla f(-2, 1)$ points directly to the left (in the $(-1, 0)$ direction).

(d) Suppose you know $(1, 0)$ is a critical point. Would you guess it's a local maximum, a local minimum, or neither? Explain your answer.

You would guess that $(1, 0)$ is a local minimum because the function values of f are increasing as you move away from $(1, 0)$.

4. Consider $f(x, y) = xe^{x^2+y^2}$ and let $(x_0, y_0) = (1, 1)$.

(a) Compute $\nabla f(1, 1)$.

First note that

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (e^{x^2+y^2} + 2x^2e^{x^2+y^2}, 2ye^{x^2+y^2}) = e^{x^2+y^2}(1 + 2x^2, 2y).$$

Evaluating this at $(1, 1)$, we get $\nabla f(1, 1) = (3e, 2e)$.

(b) Find the directional derivative $\nabla f \cdot \vec{u}(1, 1)$, where $\vec{u} = (1/2, \sqrt{3}/2)$.

$$\nabla f \cdot \vec{u}(1, 1) = (\nabla f(1, 1)) \cdot \vec{u} = (3e, 2e) \cdot (1/2, \sqrt{3}/2) = \left(\frac{3e}{2}, e\sqrt{3}\right)$$

(c) What is the direction of steepest increase for f , starting at $(1, 1)$. (Be sure to write down a unit vector!)

The direction of steepest increase is always $\nabla f/|\nabla f|$ (think about directional derivatives), which in this case is $(1/\sqrt{13})(3, 2)$.

(d) Notice $f(1, 1) = e$. What is the equation of the tangent line to the level set $\{f = e\}$, at the point $(1, 1)$?

Recall that ∇f is always perpendicular to level curves. Since the lines parallel to $\nabla f(1, 1)$ have slope $2/3$, this means the tangent to the level curve has slope $-3/2$. Thus the tangent line has the equation

$$y - 1 = -\frac{3}{2}(x - 1) \Leftrightarrow y = -\frac{3}{2}x + \frac{1}{2}.$$

5. Consider $f(x, y) = \cos x \cos y$.

(a) Classify all the critical points of f .

First we find the gradient and second partial derivatives of f :

$$\nabla f = (-\sin x \cos y, -\cos x \sin y), \quad \frac{\partial^2 f}{\partial x^2} = -\cos x \cos y = \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \sin x \sin y.$$

The critical points occur when $\nabla f = (0, 0)$, so we must have $\sin x \cos y = 0 = \cos x \sin y$. There are two possibilities: either $\sin x = 0$, which forces $\sin y = 0$, or $\cos y = 0$, which forces $\cos x = 0$. In the first case, we must have

$$(x, y) = (x_n, y_m) = (n\pi, m\pi)$$

for some integers n and m . In the second case we must have

$$(x, y) = (\hat{x}_n, \hat{y}_m) = \left(\frac{(2n+1)\pi}{2}, \frac{(2m+1)\pi}{2}\right).$$

Next we check the discriminant:

$$D = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 = \cos^2 x \cos^2 y - \sin^2 x \sin^2 y.$$

In the first case, we plug in $(x_n, y_m) = (n\pi, m\pi)$ to get $D = 1$ and $\partial^2 f / \partial x^2 = -\cos(n\pi) \cos(m\pi) = (-1)^{n+m+1}$. So this point is a local maximum for $n + m$ even and a local minimum for $n + m$ odd. In the second case, we plug in $(\hat{x}_n, \hat{y}_m) = ((2n + 1)\pi/2, (2m + 1)\pi/2)$ to get $D = -1$, so these points are all saddle points.

- (b) Find the absolute maximum and minimum of f on the square $\pi/4 \leq x \leq 3\pi/4$, $\pi/4 \leq y \leq 3\pi/4$.

Observe that the only critical point in this square is $(\pi/2, \pi/2)$, which is a saddle point. Thus the maximum of f must occur on the boundary of the square. Plug in $x = \pi/4$ to get

$$g_1(y) = f(\pi/4, y) = \frac{\cos y}{\sqrt{2}}, \quad \pi/4 \leq y \leq 3\pi/4.$$

The derivative of this function is $g_1' = -\sin(y)/\sqrt{2} < 0$ on $\pi/4 \leq y \leq 3\pi/4$. Thus g_1 is maximized at $y = \pi/4$, with a maximum value of $1/2$. Next plug in $x = 3\pi/4$ to get

$$g_2(y) = f(3\pi/4, y) = -\frac{\cos y}{\sqrt{2}}, \quad \pi/4 \leq y \leq 3\pi/4.$$

We can use symmetry of \cos . Indeed, $g_2(y) = -g_1(\pi - y)$, so the maximum value of $1/2$ occurs at $y = 3\pi/4$. Next we plug in $y = \pi/4$ to get

$$h_1(x) = f(x, \pi/4) = \frac{\cos x}{\sqrt{2}}, \quad \pi/4 \leq x \leq 3\pi/4.$$

The derivative of h_1 is $h_1' = -\sin(x)/\sqrt{2} < 0$ on $[\pi/4, 3\pi/4]$, so the maximum occurs at $x = \pi/4$, with a maximum value of $1/2$. (This is the same maximum we found earlier.) Finally, we plug in $y = 3\pi/4$ to get

$$h_2(x) = f(x, 3\pi/4) = -\frac{\cos x}{\sqrt{2}}, \quad \pi/4 \leq x \leq 3\pi/4.$$

Again, we use symmetry: $h_2(x) = -h_1(\pi - x)$, so the maximum of $1/2$ occurs at $x = 3\pi/4$. We conclude that the maximum value of f is $1/2$, which occurs at $(\pi/4, \pi/4)$ and $(3\pi/4, 3\pi/4)$. A similar analysis shows that the minimum value of $-1/2$ occurs at $(\pi/4, 3\pi/4)$ and $(3\pi/4, \pi/4)$.

6. Consider the composition $F(t) = f(x(t), y(t))$, where f is a function of the two variables x and y , while x and y are both functions of t .

- (a) Suppose $x(1) = 0$, $y(1) = 2$, $x'(1) = 2$, $y'(1) = -1$, $\frac{\partial f}{\partial x}(0, 2) = 5$, and $\frac{\partial f}{\partial y}(0, 2) = -3$. Compute $F'(1)$.

$$F'(1) = \frac{\partial f}{\partial x}(x(1), y(1))x'(1) + \frac{\partial f}{\partial y}(x(1), y(1))y'(1) = 5 \cdot 2 + (-3) \cdot (-1) = 13$$

- (b) If $x'(0) = 0$ and $y'(0) = 0$, is it true that $F'(0) = 0$? Explain your answer.

By the chain rule,

$$F' = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y'.$$

Thus if x' and y' are both zero then the sum is zero. So the answer is "yes."

- (c) If $F'(3) = 0$, is it true that $x'(3) = 0$ and $y'(3) = 0$? Explain your answer.

In this case, the answer is "no" because we can have cancellation. For instance, take $f(x, y) = x^2 - y^2$ and $(x(t), y(t)) = (t, t)$.

7. Compute the slope of the tangent line to the hyperbola $x^2 - y^2 = 1$ at the point $(2, \sqrt{3})$.

Notice that the hyperbola is given as a level set of the function $f(x, y) = x^2 - y^2$, so its tangents are perpendicular to ∇f . We compute that

$$\nabla f(2, \sqrt{3}) = (2x, -2y)|_{(2, \sqrt{3})} = (4, -2\sqrt{3}).$$

Thus the tangent to the hyperbola has slope $2/\sqrt{3}$.

8. (a) Evaluate $\int_0^1 \int_0^1 [x^2y + yx^3] dx dy$.

$$\int_0^1 \int_0^1 [x^2y + yx^3] dx dy = \int_0^1 \left[\frac{1}{3}x^3y + \frac{1}{4}x^4y \right]_{x=0}^{x=1} dy = \int_0^1 \frac{7}{12}y dy = \frac{7}{24}$$

(b) Evaluate $\int_{x^2+y^2 \leq 1} [e^{x^2+y^2}] dx dy$. (Hint: try polar coordinates.)

In polar coordinates, this integral is

$$\int_0^{2\pi} \int_0^1 r e^{r^2} dr d\theta = 2\pi \int_0^1 e^u du = \pi(e-1).$$

(c) Evaluate $\int_D [xy - x^2 y^2] dx dy$, where D is the ice-cream cone shaped region bounded by $y = x-1$, $y = -x-1$, and $x^2 + y^2 = 1$. (It might help to draw a picture of D).

9. Find the minimum of $f = y^3 - 3yx^2$ on the ellipse $x^2 + 4y^2 \leq 1$.

We first find interior critical points:

$$(0, 0) = \nabla f = (-6xy, 3y^2 - 3x^2) \Leftrightarrow (x, y) = (0, 0).$$

However, one can see that $(0, 0)$ is neither a local max nor a local min. (The easiest way to see this is to look at the zero set. This divides the plane into six wedges all meeting at $(0, 0)$, and the sign of f alternates as one travels from wedge to wedge.) So we look for critical points on the boundary using Lagrange multipliers. The boundary is given by $g(x, y) = x^2 + 4y^2 = 1$, so we set

$$\nabla f = \lambda \nabla g, \quad g = 1,$$

which is the system of equations

$$-6xy = 2\lambda x, \quad 3y^2 - 3x^2 = 8\lambda y, \quad x^2 + 4y^2 = 1.$$

If $x \neq 0$, the first equation gives $\lambda = -3y$. Plugging this into the second equation, we get $3y^2 - 3x^2 = -24y^2$, which we rearrange as $-x^2 + 9y^2 = 0$. Combine this with the last equation, we get $13y^2 = 1$, or $y = \pm 1/\sqrt{13}$, which also gives us $x = \pm 3/\sqrt{13}$. Finally, if $x = 0$ we must have $y = \pm 1/2$. This gives us 6 points to check:

$$\begin{aligned} f(0, 1/2) &= 1/8, & f(0, -1/2) &= -1/8, \\ f(3/\sqrt{13}, 1/\sqrt{13}) &= -26/\sqrt{169}, & f(3/\sqrt{13}, -1/\sqrt{13}) &= 26/\sqrt{169}, \\ f(-3\sqrt{13}, 1\sqrt{13}) &= -26/\sqrt{169}, & f(-3\sqrt{13}, -1\sqrt{13}) &= 26/\sqrt{169}. \end{aligned}$$

So we see that the minimum value is $-26/\sqrt{169}$, which occurs at $(3/\sqrt{13}, 1/\sqrt{13})$ and $(-3/\sqrt{13}, 1/\sqrt{13})$.