Solutions to the Practice Problems

Math 210

Feb. 11, 2005

1. Given the following pairs of vectors $\vec{u}$ and $\vec{v}$, find the angle θ between them and compute the cross product $\vec{u} \times \vec{v}$.

   (a) $\vec{u} = (1, 2, 3)$, $\vec{v} = (-2, 1, 0)$
   The angle θ is given by
   $$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = 0,$$
   so the angle is $\theta = \pi/2$. Next we compute the cross product:
   $$\vec{u} \times \vec{v} = (0 - 3, -6 - 0, 1 + 4) = (-3, -6, 5).$$

   (b) $\vec{u} = (1, 0, 2)$, $\vec{v} = (2, 1, 0)$
   The angle θ is given by
   $$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{2}{5},$$
   so the angle is $\theta = \cos^{-1}(2/5) \simeq 1.16$. Next we compute the cross product:
   $$\vec{u} \times \vec{v} = (0 - 2, 4 - 0, 1 - 0) = (-2, 4, 1).$$

   (c) $\vec{u} = (1, 1, 0)$, $\vec{v} = (1, 0, 1)$
   The angle θ is given by
   $$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{1}{5},$$
   so the angle is $\theta = \cos^{-1}(1/5) \simeq 1.37$. Next we compute the cross product:
   $$\vec{u} \times \vec{v} = (1 - 0, 0 - 1, 0 - 1) = (1, -1, -1).$$

2. Consider the plane Π_1 containing $p = (2, 1, 3)$, with normal vector $\vec{n} = (-1, 2, 0)$.

   (a) Write down the linear equation any point (x, y, z) in this plane must satisfy.
   The linear equation of Π_1 is given by
   $$0 = \vec{n} \cdot ((x, y, z) - p) = (-1, 2, 0) \cdot (x - 2, y - 1, z - 3) = -x + 2y.$$

   (b) Find the angle between the plane Π_1 and the plane Π_2 determined by $x - y = 2$.
   The angle between Π_1 and Π_2 is the same as the angle between the two normal vectors $\vec{n}$ and $\vec{m} = (1, -1, 0)$. This angle θ is given by
   $$\cos\theta = \frac{\vec{n} \cdot \vec{m}}{\|\vec{n}\|\|\vec{m}\|} = -\frac{3}{\sqrt{10}},$$
   so $\theta = \cos^{-1}(-3/\sqrt{10}) \simeq= 2.82$

   (c) Parameterize the line l which is the intersection of Π_1 and Π_2.
   We can parameterize l as $l(t) = q + \vec{v}t$, where q is a point common to both planes Π_1 and Π_2 and $\vec{v}$ is lies in the same direction as l. We could compute $\vec{v}$ by computing the cross product $\vec{n} \times \vec{m}$, but in this case there's an easy trick: observe that both $\vec{n}$ and $\vec{m}$ lie in the $x - y$ plane, so they are both perpendicular to the vector $\vec{v} = (0, 0, 1)$.
   Now we have to find a point q in both Π_1 and Π_2. The coordinates (x, y, z) of q will satisfy the system
   $$x - y = 2 \qquad -x + 2y = 0.$$
   Simultaneously solving for x and y we see $x = 4, y = 2$. What is z? It can be anything. One can see that from the fact that l is parallel to the z-axis. We choose $z = 0$, so $q = (4, 2, 0)$ and
   $$l(t) = (4, 2, 0) + t(0, 0, 1) = (4, 2, t).$$

(d) Find the distance between the plane Π_1 and the point $q = (3, 3, 3)$.
We can compute the distance by computing the orthogonal projection of $q-p = (3,3,3)-(2,1,3) = (1,2,0)$ onto $\vec{n} = (-1, 2, 0)$. This orthogonal projection is

$$\vec{a} = (\cos\theta)|q-p|\frac{\vec{n}}{|\vec{n}|} = \frac{(q-p)\cdot\vec{n}}{|\vec{n}|}\frac{\vec{n}}{|\vec{n}|} = \frac{3}{5}(-1,2,0).$$

The distance is the length of this orthogonal projection

$$d = |\vec{a}| = \frac{3}{5}\sqrt{1+4} = \frac{3}{\sqrt{5}}.$$

3. Consider the vectors $\vec{u} = (1, 2, 1)$ and $\vec{v} = (0, 1, -1)$.

(a) Explain why all the planes parallel to both $\vec{u}$ and $\vec{v}$ will have the same normal vectors (up to scaling).
All parallel planes have the same normal vector (up to scaling), because any plane is given by $\vec{n}\cdot((x,y,z)-(x_0,y_0,z_0))$, for some choice of of normal vector $\vec{n}$ and point (x_0, y_0, z_0) on the plane. Changing which plane you pick in a family of parallel planes amounts to vearying (x_0, y_0, z_0) (in the direction of $\vec{n}$). Thus they all have the same normal vector. Another way to think of this is: the angle between two planes is the angle between the two normal vectors. So two planes are parallel precisely when their normal vectors are parallel, which is another way of saying the normals are the same up the scaling.

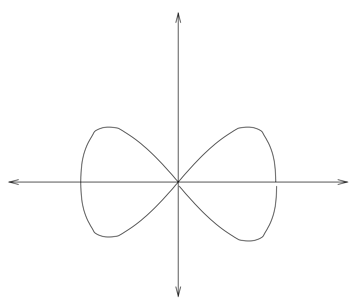
(b) Are these planes parallel to the plane given by $x + y + z = 2$? Explain your answer.
The normal vector to the plane spanned by $\vec{u}$ and $\vec{v}$ is

$$\vec{n} = \vec{u} \times \vec{v} = (-3, 1, 1),$$

while the plane $x + y + z = 3$ has the normal vector $\vec{m} = (1, 1, 1)$. These normal vectors are not parallel, so the two planes are not parallel.

4. Consider the plane curve given by $c(t) = (\cos(t), \sin(2t))$, for $0 \le t \le 2\pi$.

(a) Sketch this curve.
The curve c is a sideways figure 8, with the twist in the 8 at the origin. Here is a sketch.



(b) Set up, but do not evaluate, the integral to compute the arclength of c.
length $= \int_0^{2\pi} |c'(t)|dt = \int_0^{2\pi} \sqrt{\sin^2(t) + 4\cos^2(2t)}dt.$

(c) Notice c is periodic ($c(0) = c(2\pi)$). Is c a simple closed curve? In othre words, are the t parameters 0 and 2π the only times c crosses itself?
No, c crosses itself at the origin, which corresponds to $t = \pi/2$ and $t = 3\pi/2$.

(d) Find the area enclosed by c.
Notice that c has symmetry: it is preserved under reflection through the x- and y-axes. Thus we only need to compute the area in the first quadrant, which corresponds to $0 \le t \le \pi/2$:

$$\text{area} = 4\int_0^{\pi/2} y(t)dt = 4\int_0^{\pi/2} \sin(2t)dt = -2\,\cos(2t)|_0^{\pi/2} = 4.$$

5. (a) Consider the right circular cone C, with vertex at $(0, 0, 0)$, and slope 1. In other words, the cone C is what you get when you rotate the line $y = z$ in the $y - z$ plane about the z-axis. Write C in cylindrical coordinates.
For a given height z, the distance from the z-axis is also z and the angle can be anything. So in cylindrical coordinates, this is $r = z$, with θ and z unrestricted.

(b) Write the part of the shell $1 \le x^2 + y^2 + z^2 \le 4$ lying in the $x < 0, y > 0, z < 0$ octant in spherical coordinates.

First, $x^2 + y^2 + z^2 = \rho^2$, so we have $1 \le \rho \le 2$. Next we have to find bounds on the angles. The quadrant $x < 0$, $y > 0$ in the $x - y$ plane corresponds to $\pi/2 \le \theta \le \pi$. Also, $z < 0$ means we restrict $\pi/2 \le \phi \le \pi$. So this region is

$$1 \le \rho \le 2, \quad \pi/2 \le \theta \le \pi, \quad \pi/2 \le \phi \le \pi.$$

6. Consider the space curve $c(t) = (\cos(t), \sin(t), t)$.

(a) Is the velocity vector ever tangent to the x-axis?

The velocity is

$$c' = \frac{d}{dt}(\cos t, \sin t, t) = (-\sin t, \cos t, 1).$$

In order for c' to be tangent to the x-axis, its last two components have to vanish. However, the last component is 1, so this never happens.

(b) Verify the Fundamental Theorem of Calculus by checking

$$c(2\pi) - c(0) = \int_0^{2\pi} c'(t)dt.$$

$$c(2\pi) - c(0) = (\cos(2\pi), \sin(2\pi), 2\pi) - (\cos(0), \sin(0), 0) = (1, 0, 2\pi) - (1, 0, 0) = (0, 0, 2\pi).$$

Meanwhile,

$$\int_0^{2\pi} c'(t)dt = \int_0^{2\pi} (-\sin t, \cos t, 1)dt = \left(\cos t|_0^{2\pi}, \sin t|_0^{2\pi}, t|_0^{2\pi}\right) = (0, 0, 2\pi).$$

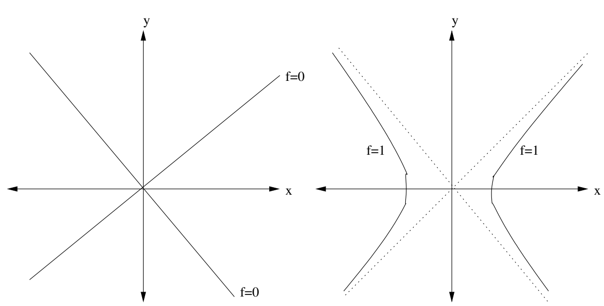
(c) Set up, but do not evaluate, the integral to compute the arclength of c for $0 \le t \le 2\pi$.

length $= \int_0^{2\pi} |c'(t)|dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}dt = \int_0^{2\pi} \sqrt{2}dt = 2\pi\sqrt{2}$.

7. Consider the function $f(x, y) = x^2 - y^2$.

(a) Sketch the level sets $f = 0$ and $f = 1$.

The level set $f = 0$ is given by $x^2 = y^2$, or $x = \pm y$. This is the union of two lines, crossing at right angles at the origin. The level set $f = 1$ is the hyperbola $x^2 - y^2 = 1$. Here is a sketch.



(b) Does f have an upper bound? How about a lower bound?

f has neither an upper bound nor a lower bound. Indeed, by setting $y = 0$, we see that $f(x, 0) = x^2$, which can be as large and positive as you please. Similarly, $f(0, y) = -y^2$, which can be as large and negative as you please.

(c) Compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$.

We have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 - y^2) = 2x$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2) = -2y.$$

(d) Is the tangent plane to the graph of f ever parallel to the $x-y$ plane?
The tangent plane has normal vector

$$\vec{n} = (1, 0, \partial f/\partial x) \times (0, 1, \partial f/\partial y) = (1, 0, 2x) \times (0, 1, -2y) = (-2x, 2y, 1).$$

Also, the $x-y$ plane has normal $(0, 0, 1)$, which is parallel to $\vec{n}$ (in fact, equal to $\vec{n}$) precisely when $x = y = 0$.

8. Explain why the tangent plane to the graph of a function $f(x, y)$ cannot ever be parallel to the $x-z$ or $y-z$ planes, provided f has continuous partial derivatives.

The tangent plane to the graph of f is spanned by $(1, 0, \partial f/\partial x)$ and $(0, 1, \partial f/\partial y)$, so it has the normal vector

$$\vec{n} = (1, 0, \partial f/\partial x) \times (0, 1, \partial f/\partial y) = (-\partial f/\partial x, -\partial f/\partial y, 1).$$

However, the normal vector to the $y-z$ plane is $(1, 0, 0)$, which cannot ever be parallel to $(-\partial f/\partial x, -\partial f/\partial y, 1)$. Similar reasoning applies to the $x-z$ plane.