Practice Problems

Math 210

November 27, 2005

These problems are not in any particular order. The exam will be shorter. The last 8 problems form the final exam from the last time I taught this course.

1. For each of the following pairs of vectors, compute $\vec{u} \cdot \vec{v}$, $\vec{u} \times \vec{v}$ and the orthogonal projection of $\vec{u}$ onto $\vec{v}$.
   (a) $\vec{u} = (1, 0, 1), \vec{v} = (1, 1, 0)$
   (b) $\vec{u} = (0, 1, 0), \vec{v} = (0, 1, -1)$
2. Let Π_1 be the plane through $(1, 1, 1)$ with normal vector $\vec{n_1} = (1, -1, 0)$, and let Π_2 be the plane defined by $x + y + z = 1$.
   (a) Write down a linear equation for Π_1.
   (b) Find the cosine angle between Π_1 and Π_2.
   (c) Parameterize the line l of intersection between Π_1 and Π_2.
   (d) Find the distance between Π_2 and $(1, 1, 1)$.
3. Consider the curve $c(t) = (\cos(2t), \sin(t))$ for $0 \le t \le 2\pi$.
   (a) Sketch this curve.
   (b) Write down the tangent line to c at the point $(-1, 1)$. (This point corresponds to the parameter value $t = \pi/2$).
   (c) Is the tangent line to this curve ever parallel to the line $y = -x$? Be sure to explain your answer.
   (d) Set up, but do not evaluate, the integral to compute the arclenth of this curve.
4. Consider the function
   $$f(x, y) = \int_y^{e^x} \sqrt{1 + t^2}dt.$$
   (a) Compute the partial derivatives of f.
   (b) Does f have any critical points? Be sure to explain your answer.
5. Consider the function $f(x, y) = x^2y - xy^3$.
   (a) Does f have an upper or lower bound? Explain your answer.
   (b) Compute the partial derivatives of f.
   (c) Compute the directional derivative of f in the $(1/\sqrt{2}, -1/\sqrt{2})$ direction, at the point $(1, 1)$.
   (d) Find the direction of steepest ascent for f, starting at $(1, 1)$. Make sure to write down a unit vector.
   (e) Notice that $f(1, 1) = 0$. Write down the equation of the tangent line to the $f = 0$ level set, at the point $(1, 1)$.
6. Consider the function
   $$f(x, y) = e^{xy^3 - x^2}.$$
   (a) Find and classify all the critical points of f.
   (b) Find the absolute minimum of f restricted to the square $0 \le x \le 1$, $0 \le y \le 1$.
7. Recall that two tangent directions to the graph of f are given by the vectors
   $$(1, 0, \frac{\partial f}{\partial x}), \qquad (0, 1, \frac{\partial f}{\partial y}).$$
   (a) Compute a normal vector for the graph.
   (b) Can the tangent plane of the graph ever be parallel to the $x - z$ plane? Explain your answer.
8. Find the absolute maximum and minimum of the function $f = xy$ on the ellipse $4x^2 + y^2 = 4$.
9. Evaluate $\int_D \sqrt{4 - x^2 - y^2}dA$ where $D = \{(x, y) \mid 1 \le x^2 + y^2 \le 4, y > 0$.

10. Set up, but do not evaluate $\int_D xe^{x^2y}dA$, where D is the region bounded by the curves $y = x^3$ and $y = x$. (Be careful of signs.)

11. Evaluate $\int\int_D \sqrt{1-x^2}dA$ where D is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

12. Consider the domain $D := \{(x,y) \mid |x+y| \leq 1\}$.

    (a) Sketch D.

    (b) If we change variables by $u = \frac{1}{\sqrt{2}}(x+y), v = \frac{1}{\sqrt{2}}(-x+y)$, what is D in the u,v coordinate system?

    (c) Set up, but do not evaluate, the integral $\int\int_D \cos(\pi x - \pi y)dA$ in the (u,v) coordiante system.

13. Consider the vector field $\vec{F} = (-y, x)$.

    (a) Sketch $\vec{F}$.

    (b) Compute $\int_\gamma \vec{F} \cdot d\vec{s}$ where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$.

14. Consider the vector field $\vec{F} = (2xe^{x^2+y^2}, 2ye^{x^2+y^2} + \cos y)$.

    (a) Show that $\vec{F} = \nabla f$ for some f.

    (b) Find a function f such that $\vec{F} = \nabla f$.

    (c) Compute $\int_\gamma \vec{F} \cdot d\vec{s}$, where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi/2$. (Hint: you don't need to actually do an integral.)

15. Consider $\vec{F} = (-y + x^2 - y\cos(xy), x - y^3 - x\cos(xy))$.

    (a) Is $\vec{F} = \nabla f$ for some f? Be sure to explain your answer.

    (b) Compute $\int_\gamma \vec{F} \cdot d\vec{s}$. (Hint: don't actually compute the line integral.)

16. Consider the vector field $\vec{F} = (z + x^3 - yze^{xyz}, y - xze^{xyz}, -x + z^2 - xye^{xyyz})$.

    (a) Compute $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$.

    (b) Is $\vec{F} = \nabla f$ for some f? Be sure to explain your answer.

    (c) Compute $\int\int_\Sigma \nabla \times \vec{F} \cdot \vec{n}dA$, where Σ is the upper unit hemisphere, centered at $(0,0,0)$, with the outward unit normal.

17. Consider the surface $\vec{r}(u,v) = (u\cos(v), u\sin(v), v)$.

    (a) Compute the tangent vectors $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$.

    (b) Verify that this is a good parameterization, by checking that $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$ are never parallel.

    (c) Find the equation of the tangent plane to $\vec{r}$ for the parameter values $u = 1, v = \pi$.

    (d) Is the tangent plane ever parallel to the $x - y$ plane? Be sure to explain your answer.

    (e) What is this surface? Can you draw a sketch of it? (Hint: fix a value of u, for instance $u = 1$ or $u = 0$, and draw the resulting curve.)

18. Compute $\int\int_\Sigma \nabla \times \vec{F} \cdot \vec{n}dA$, where $\vec{F} = (-y, x, 0)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal.

19. Compute $\int\int_\Sigma \vec{F} \cdot \vec{n}dA$, where $\vec{F} = (x + yz - \cos y, y - e^{xz} + z^2, z - x\cos(x^2y))$ and Σ is the unit sphere (centered at the origin) with the outward unit normal.

20. Compute $\int\int_\Sigma \vec{F} \cdot \vec{n}dA$, where $\vec{F} = (x + ze^{y^2+z}, y - z\cos(x+z^2), z)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal. (Hint: what is $\vec{F}$ restricted to the plane $z = 0$?)

21. Let Π_1 be the plane through $(1,0,1)$ with normal $\vec{n_1} = (1,-1,0)$, and let Π_2 be the plane given by $x = y + z$.

(a) (3 points) Find $\cos\theta$, where θ is the angle between Π_1 and Π_2.

(b) (4 points) Find a linear equation for the plane Π_1.

(c) (4 points) Parameterize the line l of intersection between Π_1 and Π_2.

(d) (4 points) Find the distance between $(1,1,2)$ and Π_2.

22. Consider $f(x,y) = x^2y - xy - x^2 - 6y - x + 6$.

(a) (5 points) Verify that the only critical points of f are $(3, 7/5)$ and $(-2, 3/5)$.

(b) (5 points) Classify these critical points as local maxima, local minima, saddle points, or none of the above.

(c) (5 points) Observe that $f(1,1) = -2$. Write down an equation for the tangent line to the level set $\{f = -2\}$, at the point $(1,1)$.

23. Recall that two tangent directions to the graph of a function $f(x,y)$ are

$$(1, 0, \frac{\partial f}{\partial x}), \qquad (0, 1, \frac{\partial f}{\partial y}).$$

(a) (5 points) Write down a normal vector to the tangent plane of the graph of f.

(b) (5 points) Is it ever possible that the tangent plane to the graph of f is parallel to the plane $x + y = 0$? Be sure to explain your answer.

24. Consider the composition $g(t) = f(x(t), y(t))$, where $x(1) = 1$, $x'(1) = -1$, $y(1) = 2$, $y'(1) = 3$, and $\nabla f(1,2) = (3,-2)$.

(a) (3 points) What is $g'(1)$?

(b) (4 points) Suppose $x'(2) = 0$ and $y'(2) = 0$. Is it necessarily true that $g'(2) = 0$? Explain your answer.

(c) (3 points) Suppose $g'(-1) = 0$. Is is necessarily true that $x'(-1) = 0$ and $y'(-1) = 0$? Explain your answer.

25. (a) (5 points) Evaluate $\int\int_D e^{x^2+y^2} dA$, where D is the domain $\{(x,y) \mid 1 \le x^2 + y^2 \le 4, x \le 0\}$.

(b) (5 points) Set up, but **do not evaluate** the integral $\int\int_D e^{x^2 y} \sin(y^3 + x) dA$, where D is the region bounded by the line $y = 1 + x$ and the curve $y = x^2$. (It might help to draw a picture.)

26. Consider the vector field $\vec{F} = (y\cos(xy), x\cos(xy) + y^2)$ in the plane.

(a) (5 points) Verify that $\vec{F}$ is the gradient of a function.

(b) (5 points) Find a function f such that $\vec{F} = \nabla f$.

(c) (5 points) Evaluate $\int_\gamma \vec{F} \cdot d\vec{s}$ where $\gamma(t) = (t, 1 - t^2)$ for $0 \le t \le 1$. (Hint: you do not need to compute an integral.)

27. Consider the vector field $\vec{F} = (y + yze^{xyz}, -x + xze^{xyz}, xye^{xyz})$ in three-space.

(a) (5 points) Compute the curl of $\vec{F}$.

(b) (5 points) Is $\vec{F}$ the gradient of a function? Be sure to explain your answer.

(c) (5 points) Evaluate $\int\int_\Sigma \nabla \times \vec{F} \cdot \vec{n} dA$, where Σ is the hemisphere $\{(x,y,z) \mid x^2 + y^2 + z^2, x \ge 0\}$, oriented with the normal $\vec{n}$ pointing away from the origin. (Hint: you have to compute an integral, but not necessarily over Σ.)

28. Consider the vector field $\vec{F} = (\cos(y^2 z^{1/3}) - x + \ln y, y - e^{xz^3} + z, z + \sin(x^2 y^6)\ln(\sqrt{1 + x^2 + y^2}))$ in three-space.

(a) (5 points) Compute the divergence of $\vec{F}$.

(b) (5 points) Evaluate $\int\int_\Sigma \vec{F} \cdot \vec{n} dA$, where Σ is the unit sphere, centered at the origin, oriented with the outward normal $\vec{n}$. (Hint: think before you compute.)