Practice Problems

Math 210

November 29, 2005

1. For each of the following pairs of vectors, compute $\vec{u} \cdot \vec{v}$, $\vec{u} \times \vec{v}$ and the orthogonal projection of $\vec{u}$ onto $\vec{v}$.

   (a) $\vec{u} = (1, 0, 1), \vec{v} = (1, 1, 0)$

   $$\vec{u} \cdot \vec{v} = (1, 0, 1) \cdot (1, 1, 0) = 1 + 0 + 0 = 1$$
   $$\vec{u} \times \vec{v} = (1, 0, 1) \times (1, 1, 0) = (0 - 1, 1 - 0, 1 - 0) = (-1, 1, 1,)$$

   We'll call the projection $\text{Proj}_{\vec{v}}(\vec{u})$. To compute it, we want to compute $|\vec{u}| \cos \theta$ times a unit vector in the same direction as $\vec{v}$, where θ is the angle between $\vec{u}$ and $\vec{v}$. We have

   $$\text{Proj}_{\vec{v}}(\vec{u}) = |\vec{u}| \vec{u} \cdot \vec{v} \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{1}{2} \vec{v} = (1/2, 1/2, 0).$$

   (b) $\vec{u} = (0, 1, 0), \vec{v} = (0, 1, -1)$

   $$\vec{u} \cdot \vec{v} = 0 + 1 + 0 = 1$$
   $$\vec{u} \times \vec{v} = (0, 1, 0) \times (0, 1, -1) = (-1 - 0, 0 - 0, 0 - 0) = (-1, 0, 0)$$
   $$\text{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{1}{2} \vec{v} = (0, 1/2, -1/2)$$

2. Let Π_1 be the plane through $(1, 1, 1)$ with normal vector $\vec{n_1} = (1, -1, 0)$, and let Π_2 be the plane defined by $x + y + z = 1$.

   (a) Write down a linear equation for Π_1.
   Denote the normal to Π_1 as $\vec{n_1} = (1, -1, 0)$. Then (x, y, z) is in Π_1 if and only if

   $$((x, y, z) - (1, 1, 1)) \cdot \vec{n_1} = 0 \Leftrightarrow x = y.$$

   (b) Find the cosine angle between Π_1 and Π_2.
   The angle θ between Π_1 and Π_2 is the same as the angle between $\vec{n_1}$ and $\vec{n_2}$, where $n_2 = (1, 1, 1)$ is the normal to Π_2. Thus we have

   $$\cos \theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = 0 \Leftarrow \theta = \frac{\pi}{2}.$$

   (c) Parameterize the line l of intersection between Π_1 and Π_2.
   The direction vector of this line is

   $$\vec{v} = \vec{n_1} \times \vec{n_2} = (1, -1, 0) \times (1, 1, 1) = (-, 1, -1, 2).$$

   Next we find a common point, which amounts to finding a simultaneous solution of the following two equations:

   $$x = y, \qquad x + y + z = 1.$$

   One can check that $(1/2, 1/2, 0)$ works. Then the line l is given by

   $$l(t) = (1/2, 1/2, 0) + t(-1, -1, 2).$$

   (d) Find the distance between Π_2 and $(1, 1, 1)$.
   We first pick a base point $(1, 0, 0)$ is Π_2 (it doesn't matter which basepoint we pick). Then a displacement vector from $(1, 1, 1)$ to Π_2 is $(1, 1, 1) - (1, 0, 0) = (0, 1, 1)$, and we can compute the distance as an orthogonal projection of this displacement vector:

   $$\text{dist} = |\text{Proj}_{\vec{n_2}}(0, 1, 1)| = \frac{|(0, 1, 1) \cdot \vec{n_2}|}{|\vec{n_2}|} = \frac{2}{\sqrt{3}}.$$

3. Consider the curve $c(t) = (\cos(2t), \sin(t))$ for $0 \le t \le 2\pi$.

   (a) Sketch this curve.

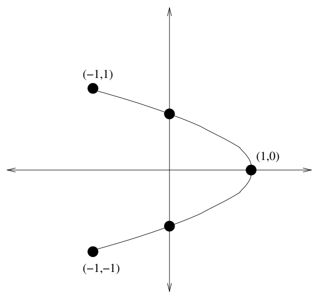

Figure 1: figure for problem 3

The marked points in this sketch correspond are $(1,0)$ (corresponding to $t = 0, \pi$), $(-1,1)$ (corresponding to $t = \pi/2$, where the curve turns around), $(-1,-1)$ (corresponding to $t = 3\pi/2$, where the curve turns around again), $(0, 1/\sqrt{2})$ (corresponding to $t = \pi/4, 3\pi/4$), and $(0, -1/\sqrt{2})$ (corresponding to $t = 5\pi/4, 7\pi/4$).

   (b) Write down the tangent line to c at the point $(-1,1)$. (This point corresponds to the parameter value $t = \pi/2$.

   We can see from the sketch that c doesn't really have a tangent line at that point, because it turns around there. We can also see this by looking at c'. First we compute the velocity vector: $c'(t) = (-2\sin(2t), \cos(t))$. Evaluating this at $t = \pi/2$, we get a direction vector for the tangent line $\vec{v} = (-2\sin(\pi), \cos(\pi/2)) = (0,0)$. This means there is no tangent line.

   (c) Is the tangent line to this curve ever parallel to the line $y = -x$? Be sure to explain your answer.

   The tanngent line is parallel to $y = -x$ precisely when $c'(t)$ is parallel to $(1,-1)$, which is a vector parallel to $y = -x$. This happens when $\cos t = 2\sin(2t) = 4\cos t \sin t$. There are two possibilities: either $\cos t = 0$, where c doesn't really have a tangent line, or $\sin t = 1/4$. The latter equation has two solutions, $t \approx .2527, 2.889$, so the tangent line to c is parallel to $y = -x$ for those values of t.

   (d) Set up, but do not evaluate, the integral to compute the arclenth of this curve.

   $\text{Length}(c) = \int_0^{2\pi} |c'(t)| dt = \int_0^{2\pi} \sqrt{4\sin^2(2t) + \cos^2 t} dt$

4. Consider the function

$$f(x,y) = \int_y^{e^x} \sqrt{1+t^2} dt.$$

   (a) Compute the partial derivatives of f.

   We use the fundamental theorem of calculus:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_y^{e^x} \sqrt{1+t^2} dt = \sqrt{1+e^{2x}} (e^x)' = e^x \sqrt{1+e^{2x}}$$

   and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_y^{e^x} \sqrt{1+t^2} dt = -\sqrt{1+y^2},$$

   (b) Does f have any critical points? Be sure to explain your answer.

   Notice that $\partial f/\partial x > 0$ and $\partial f/\partial y < 0$, so there are no points satisfying $(\partial f/\partial x, \partial f/\partial y) = (0,0)$. Thus there are no critical points.

5. Consider the function $f(x,y) = x^2 y - xy^3$.

   (a) Does f have an upper or lower bound? Explain your answer.

   Notice that if we plug in $y = x$, we get $x^3 - x^4 \to -\infty$ as $|x| \to \infty$. Also, if we plug in $y = -x$ we get $x^4 - x^3 \to \infty$ as $|x| \to \infty$. Thus f has neither a lower bound nor an upper bound.

(b) Compute the partial derivatives of f.

$$\frac{\partial f}{\partial x} = 2xy - y^3, \qquad \frac{\partial f}{\partial y} = x^2 - 3xy^2$$

(c) Compute the directional derivative of f in the $(1/\sqrt{2}, -1/\sqrt{2})$ direction, at the point $(1,1)$.
First evaluate the partial derivatives at $(1,1)$:

$$\frac{\partial f}{\partial x}(1,1) = 2 - 1 = 1, \qquad \frac{\partial f}{\partial y}(1,1) = 1 - 3 = -2.$$

Then the directional derivative is

$$\nabla_{(1/\sqrt{2},-1/\sqrt{2})} f(1,1) = (1,-2)\cdot(1/\sqrt{2},-1/\sqrt{2}) = 3/\sqrt{2}.$$

(d) Find the direction of steepest ascent for f, starting at $(1,1)$. Make sure to write down a unit vector.
The gradient $\nabla f = (\partial f/\partial x, \partial f/\partial y)$ always points in the direction of steepest ascent. The unit vector in this direction is

$$\frac{\nabla f(1,1)}{|\nabla f(1,1)|} = (1/\sqrt{5}, -2/\sqrt{5}).$$

(e) Notice that $f(1,1) = 0$. Write down the equation of the tangent line to the $f = 0$ level set, at the point $(1,1)$.
The level sets are always perpendicular to ∇f. So we choose a vector perpendicular to $\nabla f(1,1) = (1,-2)$, namely $\vec{v} = (2,1)$. Then the tangent line is parameterized as

$$l(t) = (1,1) + t(2,1),$$

or equivalently as

$$y - 1 = \frac{1}{2}(x-1).$$

6. Consider the function

$$f(x,y) = e^{xy^3 - x^2}.$$

(a) Find and classify all the critical points of f.
First we take the partial derivatives:

$$\frac{\partial f}{\partial x} = (y^3 - 2x)e^{xy^3-x^2}, \qquad \frac{\partial f}{\partial y} = 3xy^2 e^{xy^3-x^2}.$$

The critical points occur when $\nabla f = (\partial f/\partial x, \partial f/\partial y) = (0,0)$. Since the exponential is never zero, we require $y^3 - 2x = 0$ and $3xy^2 = 0$. The second equation implies either $x = 0$ or $y = 0$. The first equation implies that either x or y is zero, the so is the other one. Thus the only critical point is $(0,0)$.
Next we have to see what kind of critical point it is. We will need the second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \left.(2 + (y^3-2x)^2)e^{xy^3-x^2}\right|_{(0,0)} = 2, \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = \left.(6xy + 9x^2y^4)e^{xy^3-x^2}\right|_{(0,0)} = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \left.(3y^2 + 3xy^2(y^3-2x))e^{xy^3-x^2}\right|_{(0,0)} = 0.$$

Thus the discriminant is $D = (\partial^2 f/\partial x^2)(\partial^2 f/\partial y^2) - (\partial^2 f/\partial x \partial y)^2 = 0$, so the second derivative test is inconclusive. However, we can still classify this critical point by looking at the $f = 1$ level set. Taking a logarithm, we see that $f = 1$ is given by the equation $xy^3 = x^2$, so either $x = 0$ or $x = y^3$. These two curves cut the plane into four regions, where the exponent $xy^3 - x^2$ alternates being positive and negative. Thus $(0,0))$ is a saddle-type critical point.

(b) Find the absolute minimum of f restricted to the square $0 \le x \le 1$, $0 \le y \le 1$.
Notice that the exponential function is monotone increasing, so it will suffice to find minima of $g(x,y) = xy^3 - x^2$. At the only critical point $(0,0)$, we have $g = 0$ so $f = 1$. Also notice that if we restrict to $x = 0$, we get $g = 0$, and so $f = 1$ along this edge as well.
Next we restrict to $x = 1$, where $g = y^3 - 1$. This has a critical point at $y = 0$, with value $g = -1$ (and so $f = 1/e$). We also have $g(1,1) = 0$, and so $f(1,1) = 1$. If we restrict to $y = 0$, the we get $g = -x^2$, which has a minimum value of $g = -1$ (and so $f = 1/e$) at $(1,0)$. Finally, we restrict to $y = 1$, where $g = x - x^2$. This has a critical point at $x = 1/2$, where $g(1/2,1) = 1/4$ (and $f(1/2,1) = e^{1/4}$). Thus our candidates for minimal values of f are $f = 1$ (along the $x = 0$ and edge and at $(1,1)$), $f = 1/e$ (at $(1,0)$) and $f = e^{1/4}$ (at $(1/2,1)$). The smallest of these candidates is $f = 1/e$, which occurs at $(1,0)$.

7. Recall that two tangent directions to the graph of f are given by the vectors

$$(1, 0, \frac{\partial f}{\partial x}), \qquad (0, 1, \frac{\partial f}{\partial y}).$$

(a) Compute a normal vector for the graph.
The normal is given by

$$(1, 0, \frac{\partial f}{\partial x}) \times (0, 1, \frac{\partial f}{\partial y}) = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1).$$

(b) Can the tangent plane of the graph ever be parallel to the $x - z$ plane? Explain your answer.
The normal to $x - z = 0$ is $(0, 1, 0)$, so we'd want

$$(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1) = \lambda(0, 1, 0).$$

This is impossible because any multiple of 0 is still 0.

8. Find the absolute maximum and minimum of the function $f = xy$ on the ellipse $4x^2 + y^2 = 4$.

We use Lagrange multipliers. Letting $g = 4x^2 + y^2$, we see that the ellipse is the level set $g = 4$. Critical points of f on the ellipse then satisfy

$$\nabla f = \lambda \nabla g, \qquad g = 4,$$

where λ is some nonzero constant. We rewrite this system as

$$(y, x) = \lambda(8x, 2y), \qquad 4x^2 + y^2 = 4.$$

The first equations combine to give us $x = 16\lambda^2 x$. If $x \neq 0$, this yields $\lambda = \pm 1/4$, which we can then plug into the last equation (using $y = \pm 2x$):

$$4 = 4x^2 + y^2 = 4x^2 + (\pm 2x)^2 = 8x^2 \Leftrightarrow x = \pm\frac{1}{\sqrt{2}}.$$

This gives four critical points $\pm(1/\sqrt{2}, 1/\sqrt{2})$ and $\pm(-1/\sqrt{2}, 1/\sqrt{2})$. We still have to consider the case of $x = 0$, but in this case we also have $y = 0$, which doesn't satisfy the constraint. One can check that f achieves its maximum value of $1/2$ at $\pm(1/\sqrt{2}, 1/\sqrt{2})$, while it achieves its minimum value of $-1/2$ at $\pm(-1/\sqrt{2}, 1/\sqrt{2})$.

9. Evaluate $\int_D \sqrt{4 - x^2 - y^2} dA$ where $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y > 0$.

We will use polar coordinates. In these coordinates, D is the region $1 \leq r \leq 2$ and $f = \sqrt{4 - r^2}$. Then

$$\int\int_D f dA = \int_0^{2\pi} \int_1^2 r\sqrt{4 - r^2} dr d\theta = -\frac{1}{2}\int_0^{2\pi}\int_3^0 \sqrt{u} du d\theta = 2\pi\sqrt{3}.$$

10. Set up, but do not evaluate $\int_D xe^{x^2 y} dA$, where D is the region bounded by the curves $y = x^3$ and $y = x$. (Be careful of signs.)

We will use vertical slices (integrating with respect to y first). First notice that the curves intersect at $(0, 0)$, $(1, 1)$, and $(-1, -1)$. The region D splits into two parts: the part with $-1 < x < 0$, where $x < x^3$ and the part with $0 < x < 1$, where $x > x^3$. Thus we have

$$\int\int_D xe^{x^2 y} dA = \int_{-1}^0 \int_x^{x^3} xe^{x^2 y} dy dx + \int_0^1 \int_{x^3}^x xe^{x^2 y} dy dx.$$

11. Evaluate $\int\int_D \sqrt{1 - x^2} dA$ where D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

The triangle D is bounded by the axes and the line $x + y = 1$. We will evaluate this integral using vertical slices (integrating with respect to y first). Then

$$\begin{aligned}
\int\int_D \sqrt{1 - x^2} dA &= \int_0^1 \int_0^{1-x} \sqrt{1 - x^2} dy dx = \int_0^1 (1 - x)\sqrt{1 - x^2} dx = \int_0^1 \sqrt{1 - x^2} dx - \int_0^1 x\sqrt{1 - x^2} dx \\
&= \int_0^1 \sqrt{1 - x^2} dx - \frac{1}{2}\int_0^1 \sqrt{u} du = \int_0^1 \sqrt{1 - x^2} dx - \frac{1}{3} u^{3/2}\Big|_0^1 = \int_0^1 \sqrt{1 - x^2} dx - \frac{1}{3} \\
&= \int_0^{\pi/2} \sqrt{1 - \sin^2\theta} \cos\theta d\theta - \frac{1}{3} = \int_0^{\pi/2} \cos^2\theta d\theta - \frac{1}{3} = \frac{1}{2}\int_0^{\pi/2} (1 + 2\cos(2\theta)) d\theta - \frac{1}{3} \\
&= \frac{\pi}{4} + \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/2} - \frac{1}{3} = \frac{\pi}{4} - \frac{1}{3}
\end{aligned}$$

12. Consider the domain $D := \{(x, y) \mid |x + y| \leq 1\}$.

(a) Sketch D.
The region D is the strip between the two lines drawn below.

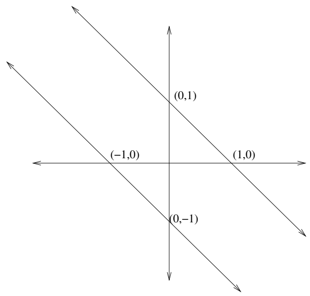

Figure 2: figure for problem 12

(b) If we change variables by $u = \frac{1}{\sqrt{2}}(x + y), v = \frac{1}{\sqrt{2}}(-x + y)$, what is D in the u, v coordinate system?
Notice that $x + y = \sqrt{2}\cdot$, so D is the domain $|u| \leq 1/\sqrt{2}$.

(c) Set up, but do not evaluate, the integral $\int\int_D \cos(\pi x - \pi y) dA$ in the (u, v) coordiante system.
Observe that

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{2}} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{1}{\sqrt{2}}.$$

So the Jacobian for the change of variables is

$$J = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \frac{1}{2} + \frac{1}{2} = 1.$$

Also, in these variables, $\cos(\pi x - \pi y) = \cos(-(\pi\sqrt{2})v) = \cos((\pi\sqrt{2})v)$. Thus the integral is

$$\int\int_D \cos(\pi x - \pi y) dA = \int_{-\infty}^{\infty} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \cos((\pi\sqrt{2})v) du dv.$$

13. Consider the vector field $\vec{F} = (-y, x)$.

(a) Sketch $\vec{F}$.

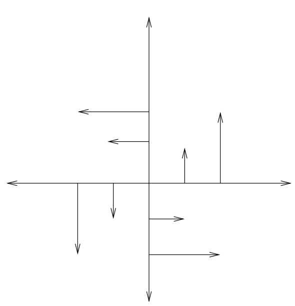

Figure 3: figure for problem 13

(b) Compute $\int_\gamma \vec{F} \cdot d\vec{s}$ where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$.
Observe that $\gamma' = (-\sin t, \cos t)$. So we have

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\gamma(t)) \cdot \gamma'(t) dt = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t) dt = \int_0^{2\pi} dt = 2\pi.$$

14. Consider the vector field $\vec{F} = (2xe^{x^2+y^2}, 2ye^{x^2+y^2} + \cos y)$.

   (a) Show that $\vec{F} = \nabla f$ for some f.
   We have to check that $\partial F_2/\partial x = \partial F_1/\partial y$. We can compute:

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(2ye^{x^2+y^2} + \cos y) = 4xye^{x^2+y^2}, \qquad \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(2xe^{x^2+y^2}) = 4xye^{x^2+y^2}.$$

   (b) Find a function f such that $\vec{F} = \nabla f$.
   We have to find an f such that

$$\frac{\partial f}{\partial x} = 2xe^{x^2+y^2}, \qquad \frac{\partial f}{\partial y} = 2ye^{x^2+y^2} + \cos y.$$

   Integrating the first equation with respect to x, we get $f = e^{x^2+y^2} + g(y)$ for some unknown function $g(y)$. The second equation then says $g'(y) = \cos y$, which we can integrate to get $g = \sin y$. Thus we have $f(x,y) = e^{x^2+y^2} + \sin y$ (up to adding a constant).

   (c) Compute $\int_\gamma \vec{F} \cdot d\vec{s}$, where $\gamma(t) = (\cos t, \sin t)$ for $0 \leq t \leq \pi/2$. (Hint: you don't need to actually do an integral.)
   The endpoints of γ are $\gamma(\pi/2) = (0,1)$ and $\gamma(0) = (1,0)$. Because $\vec{F} = \nabla f = \nabla(e^{x^2+y^2} + \sin y)$, we can use the fundamental theorem of calculus:

$$\int_\gamma \vec{F} \cdot d\vec{s} = f(0,1) - f(1,0) = (e^1 + \sin(1)) - (e^1 + \sin(0)) = \sin(1).$$

15. Consider $\vec{F} = (-y + x^2 - y\cos(xy), x - y^3 - x\cos(xy))$.

   (a) Is $\vec{F} = \nabla f$ for some f? Be sure to explain your answer.
   No. Indeed, $\partial F_2/\partial x = 1$, while $\partial F_1/\partial y = -1$. These two would have to be equal if $\vec{F} = \nabla f$.

   (b) Compute $\int_\gamma \vec{F} \cdot d\vec{s}$. (Hint: don't actually compute the line integral.)
   Forgot to say that γ is supposed to be the unit circle (centered at the origin, but that part isn't important). Then $\gamma = \partial D$, where D is the unit disc, so we can use Green's theorem:

$$\int_\gamma \vec{F} \cdot d\vec{s} = \int\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = 2\int\int_D dA = 2\pi.$$

16. Consider the vector field $\vec{F} = (z + x^3 - yze^{xyz}, y - xze^{xyz}, -x + z^2 - xye^{xyyz})$.

   (a) Compute $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$.
   First we compute the curl:

$$\nabla \times \vec{F} = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) = (0, 2, 0).$$

   Next we compute the divergence:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + 2z + 1 - (x^2y^2 + y^2z^2 + x^2z^2)e^{xyz}.$$

   (b) Is $\vec{F} = \nabla f$ for some f? Be sure to explain your answer.
   No. If $\vec{F} = \nabla f$, then $\nabla \times \vec{F} = 0$. However, we just computed that the curl of $\vec{F}$ is not 0.

   (c) Compute $\int\int_\Sigma \nabla \times \vec{F} \cdot \vec{n} dA$, where Σ is the upper unit hemisphere, centered at $(0,0,0)$, with the outward unit normal.
   Let D be the unit disc in the $x - y$ plane (with the upward normal $\vec{n} = (0,0,1)$), and observe that Σ and D have the same boundary curve γ, which is the unit circle in the $x - y$ plane. Then by Stokes theorem

$$\int\int_\Sigma \nabla \times \vec{F} \cdot \vec{n} dA = \int_\gamma \vec{F} \cdot d\vec{s} = \int\int_D \nabla \times \vec{F} \cdot \vec{n} dA = \int\int_D (0,2,0) \cdot (0,0,1) dA = 0.$$

17. Consider the surface $\vec{r}(u,v) = (u\cos(v), u\sin(v), v)$.

(a) Compute the tangent vectors $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$.

$$\frac{\partial\vec{r}}{\partial u} = (\cos v, \sin v, 0), \qquad \frac{\partial\vec{r}}{\partial v} = (-u\sin v, u\cos v, 1).$$

(b) Verify that this is a good parameterization, by checking that $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$ are never parallel.

We compute the cross product $(\partial\vec{r}/\partial u) \times (\partial\vec{r}/\partial v)$, and check that we don't get zero:

$$\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v} = (\cos v, \sin v, 0) \times (-u\sin v, u\cos v, 1) = (\sin v, -\cos v, u).$$

Notice that $\sin v$ and $\cos v$ are never simultaneously zero, so either the first or the second component of this vector is nonzero.

(c) Find the equation of the tangent plane to $\vec{r}$ for the parameter values $u = 1, v = \pi$.

First evaluate the normal vector we computed above for the parameter values $u = 1, v = \pi$. We get

$$\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v} = (0, 1, 1).$$

Next we evaluate $\vec{r}(1,\pi) = (-1, 0, \pi)$. The equation of the tangent plane is then

$$0 = (x - (-1), y - 0, z - \pi) \cdot (0, 1, 1) \Leftrightarrow y + z = \pi.$$

(d) Is the tangent plane ever parallel to the $x - y$ plane? Be sure to explain your answer.

The tangent plane to the surface is parallel to the $x - y$ plane precisely when the normal is parallel to $(0, 0, 1)$. Thus we would need

$$(\sin v, -\cos v, u) = \lambda(0, 0, 1)$$

for some $\lambda \neq 0$. However, there is no value of v such that $\sin v = 0$ and $\cos v = 0$, so this never happens, and the tangent plane is never parallel to the $x - y$ plane.

(e) What is this surface? Can you draw a sketch of it? (Hint: fix a value of u, for instance $u = 1$ or $u = 0$, and draw the resulting curve.)

This is a helicoid. If you fix a value of u, the curve you get is a helix, twisting around the z axis. (Except $u = 0$; then you just get the z axis.) The easiest way to visualize this surface is to start with the line $y = z = 0$, and raise it at a constant rate, while you twist it in the counter-clockwise direction around the z axis at the same rate. This sweeps out the surface.

18. Compute $\int\int_\Sigma \nabla \times \vec{F} \cdot \vec{n} dA$, where $\vec{F} = (-y, x, 0)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal.

There are a couple of ways to do this. We'll use Stokes theorem and parameterize $\partial\Sigma$ as $\gamma(t) = (\cos t, \sin t, 0)$ for $0 \le t \le 2\pi$. Observe that $\gamma'(t) = (-\sin t, \cos t, 0)$. Then

$$\int\int_\Sigma \nabla \times \vec{F} \cdot \vec{n} dA = \int_\gamma \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0) dt = \int_0^{2\pi} dt = 2\pi.$$

19. Compute $\int\int_\Sigma \vec{F} \cdot \vec{n} dA$, where $\vec{F} = (x + yz - \cos y, y - e^{xz} + z^2, z - x\cos(x^2 y))$ and Σ is the unit sphere (centered at the origin) with the outward unit normal.

Here we use the divergence theorem. Notice that $\Sigma = \partial B$, where B is the unit ball centered at the origin. Also, $\nabla \cdot \vec{F} = 3$. Then

$$\int\int_\Sigma \vec{F} \cdot \vec{n} dA = \int\int_{\partial B} \vec{F} \cdot \vec{n} dA = \int\int\int_B \nabla \cdot \vec{F} dV = 3\int\int\int_B dV = 4\pi.$$

20. Compute $\int\int_\Sigma \vec{F} \cdot \vec{n} dA$, where $\vec{F} = (x + ze^{y^2+z}, y - z\cos(x + z^2), z)$ and Σ is the upper unit hemisphere, centered at the origin, with the outward unit normal. (Hint: what is $\vec{F}$ restricted to the plane $z = 0$?)

Notice that $\nabla \cdot \vec{F} = 3$. We'd like to use the divergence theorem, but Σ is not a closed surface, so it's not the boundary of any solid region. However, we can add something to Σ and make the resulting union of surfaces into the boundary of a solid. In this case, we add the unit disc $D = \{(x, y, 0) \mid x^2 + y^2 \le 1\}$, with the

downward normal. Then we let $B_+ = \{(x,y,z) \mid x^2+y^2+z^2 \le 1, z \ge 0\}$ be the upper half-ball. Observe that $\partial B_+ = \Sigma + D$. Then by the divergence theorem, we have

$$\int\int_\Sigma \vec{F}\cdot\vec{n}dA + \int\int_D \vec{F}\cdot\vec{n}dA = \int\int\int_{B_+} \nabla\cdot\vec{F}dV = \int\int\int_{B_+} 3dV = 2\pi.$$

All that remains is to compute $\int\int_D \vec{F}\cdot\vec{n}dA$. We have

$$\int\int_D \vec{F}\cdot\vec{n}dA = \int_0^{2\pi}\int_0^1 (r\cos\theta, r\sin\theta, 0)\cdot(0,0,-1)rdrd\theta = 0.$$

Putting everything together, we see that

$$\int\int_\Sigma \vec{F}\cdot\vec{n}dA = \int\int\int_{B_+} \nabla\vec{F}dV - \int\int_D \vec{F}\cdot\vec{n}dA = 2\pi.$$

21. Let Π_1 be the plane through $(1,0,1)$ with normal $\vec{n_1} = (1,-1,0)$, and let Π_2 be the plane given by $x = y+z$.

(a) (3 points) Find $\cos\theta$, where θ is the angle between Π_1 and Π_2.
The angle between the planes is the same as the angle between the two normal vectors $\vec{n}_1 = (1,-1,0)$ and $\vec{n}_2 = (1,-1,-1)$. The cosine of this angle is given by

$$\cos\theta = \frac{\vec{n}_1\cdot\vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|} = \frac{(1,-1,0)\cdot(1,-1,-1)}{\|(1,-1,0)\|\|(1,-1,-1)\|} = \frac{2}{\sqrt{6}}.$$

(b) (4 points) Find a linear equation for the plane Π_1.
This linear equation is

$$\vec{n}_1\cdot(x,y,z) = \vec{n}_1\cdot(x_0,y_0,z_0) \Leftrightarrow (1,-1,0)\cdot(x,y,z) = (1,-1,0)\cdot(1,0,1) \Leftrightarrow x-y=1.$$

(c) (4 points) Parameterize the line l of intersection between Π_1 and Π_2.
We need to find two things to determine this line: a direction vector (a vector parallel to both planes) and point on both planes (a common solution to the two equations). Let's find the direction vector first. This vector $\vec{v}$ has to be perpendicular to both normals, so we choose

$$\vec{v} = \vec{n}_1\times\vec{n}_2 = (1,-1,0)\times(1,-1,-1) = (1,1,0),$$

Next we find a common solution to the two equations:

$$1 = x-y = z.$$

Notice the line of intersection is horizontal (it lies in the $z=1$ plane), so we can choose either x or y. We choose $y=0$, which forces $x=1$. Thus a point on the line is $(1,0,1)$ and a parameterization of the line is

$$l(t) = (1,0,1)+t(1,1,0).$$

(d) (4 points) Find the distance between $(1,1,2)$ and Π_2.
First we need to find a base point on the plane Π_2. The point $(1,0,1)$ we found in the previous part works just fine. Next we find the scalar projection of $\vec{PQ} = (1,1,2)-(1,0,1) = (0,1,1)$ onto the normal vector $\vec{n}_2 = (1,-1,-1)$. This is the distance we're looking for:

$$d = \frac{|\vec{PQ}\cdot\vec{n}_2|}{\|\vec{n}_2\|} = \frac{|(0,1,1)\cdot(1,-1,-1)|}{\|(1,-1,-1)\|} = \frac{2}{\sqrt{3}}.$$

22. Consider $f(x,y) = x^2y - xy - x^2 - 6y - x + 6$.

(a) (5 points) Verify that the only critical points of f are $(3,7/5)$ and $(-2,3/5)$.
We want to find the points (x,y) where $\nabla f = (\partial f/\partial x, \partial f/\partial y) = (0,0)$. These equations are

$$0 = \frac{\partial f}{\partial x} = 2xy - y - 2x - 1, \qquad 0 = \frac{\partial f}{\partial y} = x^2 - x - 6.$$

We can factor the second equation: $0 = (x-3)(x+2)$, so either $x=3$ or $x=-2$. If $x=3$ then the first equation reads $0 = 5y-7$, so $y = 7/5$. If $x=-2$ then the first equation reads $0 = -5y+3$, so $y=3/5$. Thus the only two critical points are $(3,7/5)$ and $(-2,3/5)$.

(b) (5 points) Classify these critical points as local maxima, local minima, saddle points, or none of the above.
We will need to evaluate the second partial derivatives of f to apply the second derivative test. We have

$$\frac{\partial^2 f}{\partial x^2} = 2y - 2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x - 1, \qquad \frac{\partial^2 f}{\partial y^2} = 0.$$

Thus the discriminant is

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -(2x-1)^2.$$

Evaluating D at the critical points, we see

$$D(3, 7/5) = -25 < 0, \qquad D(-2, 3/5) = -9 < 0,$$

so both critical points are saddle points.

(c) (5 points) Observe that $f(1,1) = -2$. Write down an equation for the tangent line to the level set $\{f = -2\}$, at the point $(1,1)$.
Recall that the tangent lines to level curves are perpendicular to ∇f. Here, $\nabla f(1,1) = (-2,-6)$, so the tangent line has slope $-2/6 = -1/3$. Thus the tangent line has the equation

$$y - 1 = -\frac{1}{3}(x-1) \Leftrightarrow y = -\frac{x}{3} + \frac{4}{3}.$$

23. Recall that two tangent directions to the graph of a function $f(x,y)$ are

$$(1, 0, \frac{\partial f}{\partial x}), \qquad (0, 1, \frac{\partial f}{\partial y}).$$

(a) (5 points) Write down a normal vector to the tangent plane of the graph of f.
The normal vector will be the cross product of the two tangent vectors:

$$\vec{n} = (1, 0, \frac{\partial f}{\partial x}) \times (0, 1, \frac{\partial f}{\partial y}) = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1).$$

(b) (5 points) Is it ever possible that the tangent plane to the graph of f is parallel to the plane $x + y = 0$? Be sure to explain your answer.
No. The plane $x + y = 0$ has a normal $\vec{N} = (1,1,0)$, which cannot be a scalar multiple of $\vec{n} = (-\partial f/\partial x, -\partial f/\partial y, 1)$, because the third component of $\vec{N}$ is zero, while the third component of $\vec{n}$ is nonzero.

24. Consider the composition $g(t) = f(x(t), y(t))$, where $x(1) = 1$, $x'(1) = -1$, $y(1) = 2$, $y'(1) = 3$, and $\nabla f(1,2) = (3,-2)$.

(a) (3 points) What is $g'(1)$?
By the chain rule,

$$g'(1) = \nabla f(x(1), y(1)) \cdot (x'(1), y'(1)) = \nabla f(1,2) \cdot (-1,3) = (3,-2) \cdot (-1,3) = -9.$$

(b) (4 points) Suppose $x'(2) = 0$ and $y'(2) = 0$. Is it necessarily true that $g'(2) = 0$? Explain your answer.
Yes:

$$g'(2) = \nabla f(x(2), y(2)) \cdot (x'(2), y'(2)) = \nabla f \cdot (0,0) = 0.$$

(c) (3 points) Suppose $g'(-1) = 0$. Is is necessarily true that $x'(-1) = 0$ and $y'(-1) = 0$? Explain your answer.
No. You could have $\nabla f(x(-1), y(-1)) = (0,0)$. You could also have some cancelation, such as $x'(-1) = 1, y'(-1) = 1, \nabla f = (1,1)$.

25. (a) (5 points) Evaluate $\int\int_D e^{x^2+y^2} dA$, where D is the domain $\{(x,y) \mid 1 \le x^2 + y^2 \le 4, x \le 0\}$.
We use polar coordinates: $x = r\cos\theta, y = r\sin\theta$. Then the domain is $1 \le r \le 2$ and the function is $f = e^{r^2}$. Also recall that the area element is $dA = r dr d\theta$. Then

$$\int\int_D e^{x^2+y^2} dA = \int_0^{2\pi}\int_1^2 re^{r^2} dr d\theta = 2\pi \int_1^2 re^{r^2} dr = \pi\, e^{r^2}\Big|_1^2 = \pi(e^4 - e).$$

(b) (5 points) Set up, but **do not evaluate** the integral $\int\int_D e^{x^2y}\sin(y^3+x)dA$, where D is the region bounded by the line $y = 1+x$ and the curve $y = x^2$. (It might help to draw a picture.)

The curve $y = x^2$ is a parabola opening to the up, which intersects the line $y = 1+x$ at $x^2 = x+1$, or $0 = x^2 - x - 1$. Using the quadratic formula, we get $x = (1 \pm \sqrt{5})/2$, and so $y = (3 \pm \sqrt{5})/2$. Also, for $(1-\sqrt{5})/2 \leq x \leq (1+\sqrt{5})/2$ the line lies above the parabola. We will set up the integral using vertical slices (b/c using horizontal slices we'd have to set up two integrals):

$$\int\int_D e^{x^2y}\sin(y^3+x)dA = \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} \int_{x^2}^{1+x} e^{x^2y}\sin(y^3+x)dydx.$$

26. Consider the vector field $\vec{F} = (y\cos(xy), x\cos(xy) + y^2)$ in the plane.

(a) (5 points) Verify that $\vec{F}$ is the gradient of a function.

We test that $\partial F_2/\partial x = \partial F_1 \partial y$:

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(x\cos(xy) + y^2) = \cos(xy) - xy\sin(xy), \qquad \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(y\cos(xy)) = \cos(xy) - xy\sin(xy).$$

(b) (5 points) Find a function f such that $\vec{F} = \nabla f$.

We want $\partial f/\partial x = y\cos(xy)$, so we try $f(x,y) = \sin(xy) + g(y)$ (integrating in x). Now we compare to $\partial f/\partial y$:

$$x\cos(xy) + g' = F_2 = x\cos(xy) + y^2 \Leftrightarrow g' = y^2.$$

Thus we can choose $g(y) = y^3/3$, and so

$$f = \sin(xy) + \frac{y^3}{3}.$$

(c) (5 points) Evaluate $\int_\gamma \vec{F}\cdot d\vec{s}$ where $\gamma(t) = (t, 1-t^2)$ for $0 \leq t \leq 1$. (Hint: you do not need to compute an integral.)

Note that $\gamma(0) = (0,1)$ while $\gamma(1) = (1,0)$. Then by the Fundamental Theorem of Calculus, we have

$$\int_\gamma \vec{F}\cdot d\vec{s} = f(1,0) - f(0,1) = 0 - \frac{1}{3} = -\frac{1}{3}.$$

27. Consider the vector field $\vec{F} = (y + yze^{xyz}, -x + xze^{xyz}, xye^{xyz})$ in three-space.

(a) (5 points) Compute the curl of $\vec{F}$.

We plug in the formula for the curl of a vector field and see

$$\nabla\times\vec{F} = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) = (0,0,-2).$$

(b) (5 points) Is $\vec{F}$ the gradient of a function? Be sure to explain your answer.

No. The curl of a gradient is always zero.

(c) (5 points) Evaluate $\int\int_\Sigma \nabla\times\vec{F}\cdot\vec{n}dA$, where Σ is the hemisphere $\{(x,y,z) \mid x^2+y^2+z^2, x \geq 0\}$, oriented with the normal $\vec{n}$ pointing away from the origin. (Hint: you have to compute an integral, but not necessarily over Σ.)

Notice that Σ has the same boundary as the unit disc D in the xy–plane, centered at the origin, oriented with the upward unit normal $\vec{N} = (0,0,1)$. Then we can use Stokes theorem as follows:

$$\int\int_\Sigma \nabla\times\vec{F}\cdot\vec{n}dA = \int_{\partial\Sigma}\vec{F}\cdot d\vec{s} = \int_{\partial D}\vec{F}\cdot d\vec{s} = \int\int_D \nabla\times\vec{F}\cdot\vec{N}dA = \int\int_D (0,0,-2)\cdot(0,0,1)dA = -2\int\int_D dA = -2\pi.$$

28. Consider the vector field $\vec{F} = (\cos(y^2 z^{1/3}) - x + \ln y, y - e^{xz^3} + z, z + \sin(x^2 y^6)\ln(\sqrt{1 + x^2 + y^2}))$ in three-space.

(a) (5 points) Compute the divergence of $\vec{F}$.
We use the formula for divergence:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1.$$

(b) (5 points) Evaluate $\int\int_\Sigma \vec{F} \cdot \vec{n} dA$, where Σ is the unit sphere, centered at the origin, oriented with the outward normal $\vec{n}$. (Hint: think before you compute.)
We use the divergence theorem, and the fact that the unit sphere Σ bounds the unit ball B, which has volume $4\pi/3$:

$$\int\int_\Sigma \vec{F} \cdot \vec{n} dA = \int\int\int_B \nabla \cdot \vec{F} dV = \int\int\int_B dV = \frac{4\pi}{3}.$$