Solutions to the Practice Problems

Math 210

October 18, 2005

1. Consider the function $f(x,y) = x^3y - yx^2 + 3xy$.

   (a) Write down the equation of the tangent plane to the graph of f at $(0,1)$.
   Note $f(0,1) = 0$. Also,

   $$\frac{\partial f}{\partial x} = 3x^2y - 2xy + 3y, \quad \frac{\partial f}{\partial x}(0,1) = 3, \quad \frac{\partial f}{\partial y} = x^3 - x^2 + 3x, \quad \frac{\partial f}{\partial y}(0,1) = 0.$$

   The tangent plane at $(0,1)$ is given by

   $$0 = (x-0, y-1, z-0) \cdot \left(-\frac{\partial f}{\partial x}(0,1), -\frac{\partial f}{\partial y}(0,1), 1\right) = -3x + z,$$

   which we can rearrange to read $z = 3x$.

   (b) At which points (x,y) is the tangent plane horizontal?
   We want to find points where the normal vector $(-\partial f/\partial x, -\partial f/\partial y, 1)$ is vertical, so we want $\partial f/\partial x = 0 = \partial f/\partial y$. Let's look at $\partial f/\partial y$ first:

   $$0 = \frac{\partial f}{\partial y} = x^3 - x^2 + 3x = x(x^2 - x + 3) \Leftrightarrow x = 0.$$

   Next we plug $x = 0$ into $\partial f/\partial x$ to see

   $$0 = \frac{\partial f}{\partial x}(0,y) = 3y \Leftrightarrow y = 0.$$

   Thus the only point with a horizontal tangent plane is $(0,0)$.

2. Consider

   $$f(x,y) = \int_x^y e^{t^2}\,dt.$$

   (a) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
   We use the Fundamental Theorem of Calculus:

   $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_x^y e^{t^2}\,dt = -e^{x^2}$$

   and

   $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_x^y e^{t^2}\,dt = e^{y^2}.$$

   (b) Does f have any critical points? Explain your answer.
   Critical points satisfy $\partial f/\partial x = 0 = \partial f/\partial y$. However, both partial derivatives are exponentials, which are never zero. So there are no critical points.

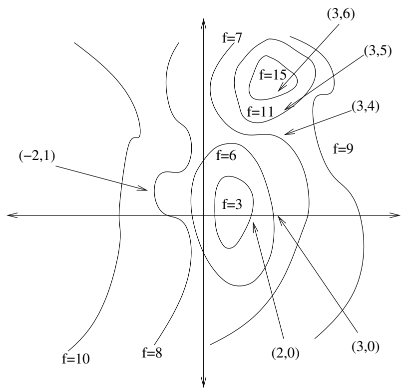
3. Let some of the level sets of the function f be given by the figure below.

   (a) Estimate $\frac{\partial f}{\partial x}(2,0)$
   Notice that $(2,0)$ is on the $f = 3$ level set, while $(3,0)$ is on the $f = 6$ level set. So

   $$\frac{\partial f}{\partial x}(2,0) \approx \frac{6-3}{3-2} = 3.$$

   (b) Estimate $\frac{\partial f}{\partial y}(3,4)$
   Notice that $(3,4)$ is on the $f = 7$ level set, while $(3,5)$ is on the $f = 11$ level set. So

   $$\frac{\partial f}{\partial y}(3,4) \approx \frac{11-7}{5-4} = 4.$$



(c) In which direction does $\nabla f(-2, 1)$ point?
Notice that $(-2, 1)$ is on the $f = 8$ level set, and this curve is vertical at $(-2, 1)$. Also, the $f = 10$ level set is to the left and the $f = 6$ and $f = 7$ level sets are to the right. Thus, because ∇f is perpendicular to level sets and points in the direction f increases, we know $\nabla f(-2, 1)$ points directly to the left (in the $(-1, 0)$ direction).

(d) Suppose you know $(1, 0)$ is a critical point. Would you guess it's a local maximum, a local minimum, or neither? Explain your answer.
You would guess that $(1, 0)$ is a local minimum because the function values of f are increasing as you move away from $(1, 0)$.

4. Consider $f(x, y) = xe^{x^2+y^2}$ and let $(x_0, y_0) = (1, 1)$.

(a) Compute $\nabla f(1, 1)$.
First note that

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (e^{x^2+y^2} + 2x^2 e^{x^2+y^2}, 2xye^{x^2+y^2}) = e^{x^2+y^2}(1 + 2x^2, 2xy).$$

Evaluating this at $(1, 1)$, we get $\nabla f(1, 1) = (3e^2, 2e^2)$.

(b) Find the directional derivative $\nabla f \cdot \vec{u}(1, 1)$, where $\vec{u} = (1/2, \sqrt{3}/2)$.
$\nabla f \cdot \vec{u}(1, 1) = (\nabla f(1, 1)) \cdot \vec{u} = (3e^2, 2e^2) \cdot (1/2, \sqrt{3}/2) = e^2(3/2 + \sqrt{3})$.

(c) What is the direction of steepest increase for f, starting at $(1, 1)$. (Be sure to write down a unit vector!)
The direction of steepest increase is always $\nabla f/|\nabla f|$ (think about directional derivatives), which is this case is $(1/\sqrt{13})(3, 2)$.

(d) Notice $f(1, 1) = e^2$. What is the equation of the tangent line to the level set $\{f = e^2\}$, at the point $(1, 1)$?
Recall that ∇f is always perpendicular to level curves. Since the lines parallel to $\nabla f(1, 1)$ have slope $2/3$, this means the tangent to the level curve has slope $-3/2$. Thus the tangent line has the equation

$$y - 1 = -\frac{3}{2}(x - 1) \Leftrightarrow y = -\frac{3}{2}x + \frac{1}{2}.$$

5. Consider $f(x, y) = \cos x \cos y$.

(a) Classify all the critical points of f.
First we find the gradient and second partial derivatives of f:

$$\nabla f = (-\sin x \cos y, -\cos x \sin y), \qquad \frac{\partial^2 f}{\partial x^2} = -\cos x \cos y = \frac{\partial^2 f}{\partial y^2}, \qquad \frac{\partial^2 f}{\partial x \partial y} = \sin x \sin y.$$

The critical points occur when $\nabla f = (0, 0)$, so we must have $\sin x \cos y = 0 = \cos x \sin y$. There are two possibilities: either $\sin x = 0$, which forces $\sin y = 0$, or $\cos y = 0$, which forces $\cos x = 0$. In the first case, we must have

$$(x, y) = (x_n, y_m) = (n\pi, m\pi)$$

for some integers n and m. In the second case we must have

$$(x, y) = (\hat{x}_n, \hat{y}_m) = (\frac{(2n + 1)\pi}{2}, \frac{(2m + 1)\pi}{2}).$$

Next we check the discriminant:

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - (\frac{\partial^2 f}{\partial y \partial x})^2 = \cos^2 x \cos^2 y - \sin^2 x \sin^2 y.$$

In the first case, we plug in $(x_n, y_m) = (n\pi, m\pi)$ to get $D = 1$ and $\partial^2 f/\partial x^2 = -\cos(n\pi)\cos(m\pi) = (-1)^{n+m+1}$. So this point is a local maximum for $n+m$ even and a local minimum for $n+m$ odd. In the second case, we plug in $(\hat{x}_n, \hat{y}_m) = ((2n+1)\pi/2, (2m+1)\pi/2)$ to get $D = -1$, so these points are all saddle points.

(b) Find the absolute maximum and minimum of f on the square $\pi/4 \le x \le 3\pi/4$, $\pi/4 \le y \le 3\pi/4$.

Observe that the only critical point in this square is $(\pi/2, \pi/2)$, which is a saddle point. Thus the maximum of f must occur on the boundary of the square. Plug in $x = \pi/4$ to get

$$g_1(y) = f(\pi/4, y) = \frac{\cos y}{\sqrt{2}}, \qquad \pi/4 \le y \le 3\pi/4.$$

The derivative of this function is $g_1' = -\sin(y)/\sqrt{2} < 0$ on $\pi/4 \le y \le 3\pi/4$. Thus g_1 is maximized at $y = \pi/4$, with a maximum value of $1/2$. Next plug in $x = 3\pi/4$ to get

$$g_2(y) = f(3\pi/4, y) = -\frac{\cos y}{\sqrt{2}}, \qquad \pi/4 \le y \le 3\pi/4.$$

We can use symmetry of cos. Indeed, $g_2(y) = -g_1(\pi - y)$, so the maximum value of $1/2$ occurs at $y = 3\pi/4$. Next we plug in $y = \pi/4$ to get

$$h_1(x) = f(x, \pi/4) = \frac{\cos x}{\sqrt{2}}, \qquad \pi/4 \le x \le 3\pi/4.$$

The derivative of h_1 is $h_1' = -\sin(x)/\sqrt{2} < 0$ on $[\pi/4, 3\pi/4]$, so the maximum occurs at $x = \pi/4$, with a maximum value of $1/2$. (This is the same maximum we found earlier.) Finally, we plug in $y = 3\pi/4$ to get

$$h_2(x) = f(x, 3\pi/4) = -\frac{\cos x}{\sqrt{2}}, \quad \pi/4 \le x \le 3\pi/4.$$

Again, we use symmetry: $h_2(x) = -h_1(\pi - x)$, so the maximum of $1/2$ occurs at $x = 3\pi/4$ We conclude that the maximum value of f is $1/2$, which occurs at $(\pi/4, \pi/4)$ and $(3\pi/4, 3\pi/4)$. A similar analysis show that the minimum value of $-1/2$ occurs at $(\pi/4, 3\pi/4)$ and $(3\pi/4, \pi/4)$.

6. Consider the composition $F(t) = f(x(t), y(t))$, where f is a function of the two variables x and y, while x and y are both functions of t.

(a) Suppose $x(1) = 0$, $y(1) = 2$, $x'(1) = 2$, $y'(1) = -1$, $\frac{\partial f}{\partial x}(0, 2) = 5$, and $\frac{\partial f}{\partial y}(0, 2) = -3$. Compute $F'(1)$.

$F'(1) = \frac{\partial f}{\partial x}(x(1), y(1))x'(1) + \frac{\partial f}{\partial y}(x(1), y(1))y'(1) = 5 \cdot 2 + (-3) \cdot (-1) = 13$

(b) If $x'(0) = 0$ and $y'(0) = 0$, is it true that $F'(0) = 0$? Explain your answer.

By the chain rule,

$$F' = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y'.$$

Thus if x' and y' are both zero then the sum is zero. So the answer is "yes."

(c) If $F'(3) = 0$, is it true that $x'(3) = 0$ and $y'(3) = 0$? Explain your answer.

In this case, the answer is "no" because we can have cancellation. For instance, take $f(x, y) = x^2 - y^2$ and $(x(t), y(t)) = (t, t)$.

7. Compute the slope of the tangent line to the hyperbola $x^2 - y^2 = 1$ at the point $(2, \sqrt{3})$.

Notice that the hyperbola is given as a level set of the function $f(x, y) = x^2 - y^2$, so its tangents are perpendicular to ∇f. We compute that

$$\nabla f(2, \sqrt{3}) = (2x, -2y)|_{(2,\sqrt{3})} = (4, -2\sqrt{3}).$$

Thus the tangent to the hyperbola has slope $2/\sqrt{3}$.

8. (a) Evaluate $\int_0^1 \int_0^1 [x^2 y + yx^3] dxdy$.

$\int_0^1 \int_0^1 [x^2 y + yx^3] dxdy = \int_0^1 [\frac{1}{3}x^3 y + \frac{1}{4}x^4 y]_{x=0}^{x=1} dy = \int_0^1 \frac{7}{12} y dy = \frac{7}{24}$

(b) Evaluate $\int_{x^2+y^2\leq 1}[e^{x^2+y^2}]dxdy$. (Hint: try polar coordiantes.)
In polar coordinates, this integral is

$$\int_0^{2\pi}\int_0^1 re^{r^2}drd\theta = \pi\int_0^1 e^u du = \pi(e-1).$$

(c) Evaluate $\int_D[xy - x^2y^2]dxdy$, where D is the ice-cream cone shaped region bounded by $y = x-1$, $y = -x-1$, and $x^2+y^2 = 1$. (It might help to draw a picture of D).
The region D looks like an ice cream cone. Notice that D is symmetric about the y–axis while xy is anti-symmetric, which means

$$\int\int_D xydA = 0.$$

Let D^+ be the part of D with $x \geq 0$. Then (because $x^2y^2 = (-x)^2y^2$), we have

$$\begin{aligned}\int\int_D xy - x^2y^2dA &= -2\int\int_{D^+} x^2y^2dA \\ &= \int_0^1\int_{x-1}^{\sqrt{1-x^2}} x^2y^2dydx = -\frac{2}{3}\int_0^1 x^2\ y^3\big|_{y=x-1}^{y=\sqrt{1-x^2}}dx \\ &= -\frac{2}{3}\int_0^1 x^2(1-x^2)^{3/2} - x^2(x-1)^3dx.\end{aligned}$$

We first evaluate $\int_0^1 x^2(x-1)^3dx$:

$$\int_0^1 x^2(x-1)^3dx = \int_0^1 x^5 - 3x^4 + 3x^3 - x^2dx = \frac{1}{6} - \frac{3}{5} + \frac{3}{4} - \frac{1}{3} = -\frac{1}{60}.$$

Next we evaluate $\int_0^1 x^2(1-x^2)^{3/2}dx$. We will first integrate by parts (with $u = x$ and $dv = x(1-x^2)^{3/2}dx$) and then do a trigonometric substitution (with $x = \sin\theta$).

$$\begin{aligned}\int_0^1 x^2(1-x^2)^{3/2}dx &= -\frac{x}{5}(1-x^2)^{3/2}\Big|_0^1 + \frac{1}{5}\int_0^1(1-x^2)^{5/2}dx = \frac{1}{5}\int_0^1(1-x^2)^{5/2}dx \\ &= \frac{1}{5}\int_{\pi/2}^0(1-\sin^2\theta)^{5/2}(\cos\theta)d\theta = \int_0^{\pi/2}\cos^6\theta d\theta = \frac{1}{5}\int_0^{\pi/2}((1/2)(1+\cos(2\theta)))^3d\theta \\ &= \frac{1}{40}\int_0^{\pi/2} 1 + 3\cos(2\theta) + 3\cos^2(2\theta) + 3\cos^3(2\theta)d\theta \\ &= \frac{\pi}{80} + \frac{3}{80}\sin(2\theta)|_0^{\pi/2} + \frac{3}{40}\int_0^{\pi/2}(1/2)(1+\cos(4\theta))d\theta + \frac{1}{40}\int_0^{\pi/2}\cos(2\theta)(1-\sin^2(2\theta))d\theta \\ &= \frac{\pi}{80} + \frac{3\pi}{160} + \frac{3}{80}\int_0^1(1-u^2)du = \frac{\pi}{80} + \frac{3\pi}{160} + \frac{1}{40} = \frac{5\pi+4}{160}.\end{aligned}$$

Putting this all together, we have

$$\int\int_D xy - x^2y^2dA = -\frac{2}{3}\cdot\frac{5\pi+4}{160} + \frac{2}{3}\cdot\frac{-1}{60} = \frac{4-5\pi}{240} - \frac{1}{90} = \frac{4-15\pi}{720}.$$

9. Find the minimum of $f = y^3 - 3yx^2$ on the ellipse $x^2 + 4y^2 \leq 1$.
We first find interior critical points:

$$(0,0) = \nabla f = (-6xy, 3y^2 - 3x^2) \Leftrightarrow (x,y) = (0,0).$$

However, one can see that $(0,0)$ is neither a local max nor a local min. (The easiest way to see this is to look at the zero set. This divides the plane into six wedges all meeting at $(0,0)$, and the sign of f alternates as one travels from wedge to wedge.) So we look for critical points on the boundary using Lagrange multipliers. The boundary is given by $g(x,y) = x^2 + 4y^2 = 1$, so we set

$$\nabla f = \lambda\nabla g, \qquad g = 1,$$

which is the system of equations

$$-6xy = 2\lambda x, \qquad 3y^2 - 3x^2 = 8\lambda y, \qquad x^2 + 4y^2 = 1.$$

If $x \neq 0$, the first equation gives $\lambda = -3y$. Plugging this into the second equation, we get $3y^2 - 3x^2 = -24y^2$, which we rearrange as $-x^2 + 9y^2 = 0$. Combine this with the last equation, we get $13y^2 = 1$, or $y = \pm 1/\sqrt{13}$, which also gives us $x = \pm 3/\sqrt{13}$. Finally, if $x = 0$ we must have $y = \pm 1/2$. This gives us 6 points to check:

$$\begin{aligned} f(0, 1/2) &= 1/8, \qquad f(0,-1/2) = -1/8, \\ f(3/\sqrt{13}, 1/\sqrt{13}) &= -26/\sqrt{169}, \qquad f(3/\sqrt{13}, -1/\sqrt{13}) = 26/\sqrt{169}, \\ f(-3\sqrt{13}, 1\sqrt{13}) &= -26/\sqrt{169}, \qquad f(-3\sqrt{13}, -1\sqrt{13}) = 26/\sqrt{169}. \end{aligned}$$

So we see that the minimum value is $-26/\sqrt{169}$, which occurs at $(3/\sqrt{13}, 1/\sqrt{13})$ and $(-3/\sqrt{13}, 1/\sqrt{13})$.

10. Consider the function

$$f(x,y) = \int_{2-y}^{x} \sqrt{1+t^2}dt.$$

(a) (5 points) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
Use the fundamental theorem of calculus for functions of one variable:

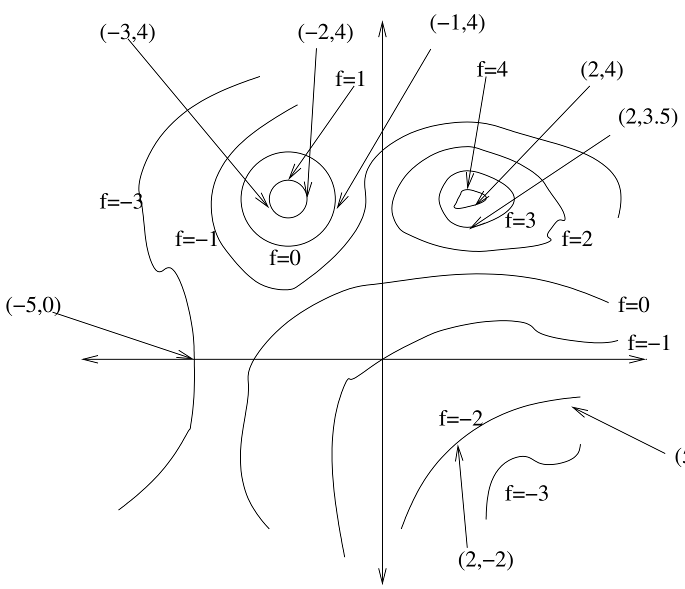
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_{2-y}^{x} \sqrt{1+t^2}dt = \sqrt{1+x^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_{2-y}^{x} \sqrt{1+t^2}dt = -\sqrt{1+(2-y)^2}(-1) = \sqrt{1+(2-y)^2}$$

(b) (5 points) Does f have any critical points? Be sure to explain your answer.
No. Both partial derivative are always positive, so there can never be a point (x_0, y_0) with $\nabla f(x_0, y_0) = (0,0)$.

11. Consider the following sketch of level curves of the function $f(x,y)$.



(a) (3 points) Estimate $\frac{\partial f}{\partial y}(2, 3.5)$.

$$\frac{\partial f}{\partial y}(2,3.5) \approx \frac{f(2,4) - f(2,3.5)}{4 - 3.5} = 2$$

(b) (4 points) In which direction does $\nabla f(-5,0)$ point? Be sure to explain your answer.
First observe that $(-5,0)$ lies on the $f = -3$ level curve, so $\nabla f(-5,0)$ must be perpendicular to that curve. At $(-5,0)$, this level curve is vertical, so ∇f points either in the $(1,0)$ direction or in the $(-1,0)$ direction (to the right or the left). Also, f increases if you move to the right, while it decreases if you move to the left. Thus $\nabla f(-5,0)$ points in the $(1,0)$ direction (to the right).

(c) (3 points) If f has a critical point at $(-2.5, 4.5)$, do you expect it to be a local minimum, a local maximum, or a saddle point? Be sure to explain your answer.
Because f decreases as you move away from $(-2.5, 4.5)$ in any direction, you would expect that f has a local maximum there.

12. Consider the function

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

(a) (5 points) Verify that the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.
Note that

$$\nabla f = (4x^3 - 4y, 4y^3 - 4x).$$

Critical points satisfy $\nabla f = (0, 0)$, which we can rewrite as

$$x^3 = y, \qquad y^3 = x.$$

The substituting the first equation in to the second, we get

$$x^9 = x \Leftrightarrow 0 = x^9 - x = x(x^8 - 1).$$

The only solutions to this equation are $x = 0, 1, -1$. If $x = 0$ we get $y = 0$, if $x = 1$ we get $y = 1$, and if $x = -1$ we get $y = -1$. Thus $(0, 0)$, $(1, 1)$ and $(-1, -1)$ are the only critical points.

(b) (5 points) Classify these critical points as local maxima, local minima, or saddle points.
First we compute second derivatives and the discriminant:

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \quad \frac{\partial^2 f}{\partial y^2} = 12y^2, \quad \frac{\partial^2 f}{\partial x \partial y} = -4, \quad D = 144x^2y^2 - 16.$$

Plugging in our critical points, we see $D(0, 0) = -16 < 0$, so $(0, 0)$ is a saddle point. Next, $D(1, 1) = 140 > 0$ and $\partial^2 f/\partial x^2(1, 1) = 12 > 0$, so $(1, 1)$ is a local minimum. Finally, $D(-1, -1) = 140 > 0$ and $\partial^2 f/\partial x^2 = 12 > 0$, so $(-1, -1)$ is also a local minimum.

13. (10 points) Find the absolute maximum of $f(x, y) = xy$ on the ellipse $g(x, y) = x^2 + 4y^2 \leq 1$.
First we look for interior critical points:

$$(0, 0) = \nabla f = (y, x) \Leftrightarrow x = 0 = y.$$

However, one can check that $D \equiv -1$ in this case, so $(0, 0)$ is a saddle point and the maximum must occur on the boundary. Next we use Lagrange multipliers:

$$\nabla f = \lambda \nabla g, \qquad g = 1,$$

which we rewrite as

$$y = 2\lambda x, \qquad x = 8\lambda y, \qquad x^2 + 4y^2 = 1.$$

Then either $x = 0$ (which implies $y = 0$, a case we've already examined), or $16\lambda^2 = 1$. We plug $\lambda = \pm 1/4$ back into our equations to get $x = \pm 2y$, so

$$1 = x^2 + 4y^2 = 4y^2 + 4y^2 \Leftrightarrow y = \pm \frac{1}{\sqrt{8}}.$$

This in turn gives $x = \pm 1/\sqrt{2}$, and so we have four candidates for maximum points:

$$(1/\sqrt{2}, \sqrt{2}), (-1/\sqrt{2}, \sqrt{2}), (1/\sqrt{2}, -\sqrt{2}), (-1/\sqrt{2}, -\sqrt{2}).$$

Testing these, we find maximal function values (of 1/4) at $\pm(1/\sqrt{2}, 1/\sqrt{2})$. The other two points are minima.

14. (a) (5 points) Where D is the square $\{1 \leq x \leq 2, -2 \leq y \leq -1\}$, evaluate

$$\int\int_D [x^2y + yx^3] dA.$$

We'll integrate with respect to x first, but in this case it doesn't make too much difference.

$$\begin{aligned}\int\int_D [x^2y + yx^3] dA &= \int_{-2}^{-1} \int_1^2 [x^2y + yx^3] dx dy = \int_{-2}^{-1} [\frac{1}{3}x^3y\Big|_1^2 + \frac{1}{4}x^4y\Big|_1^2 dy \\ &= \int_{-2}^{-1} [8y/3 + 15y/4] dy = \frac{77}{12} \int_{-2}^{-1} y dy = \frac{77}{24}\, y^2\Big|_{-2}^{-1} = -\frac{77}{8}.\end{aligned}$$

(b) (5 points) Set up, but do **not** evaluate, the integral $\int\int_D \sqrt{1+x^2+y^2}dA$, where D is the domain bounded by the curves $y = x+1$ and $x = -y^2$.

We'll integrate first with respect to x (taking horizontal slices), because it's much easier. It helps to draw the domain. It's a region bounded on the left by a slanted line and on the right by a parabola opening to the left. One can find the intersection points of these two curves by setting

$$-y^2 = x = y - 1 \Leftrightarrow 0 = y^2 + y - 1 \Leftrightarrow y = \frac{-1 \pm \sqrt{5}}{2}.$$

This gives the upper and lower limits of the y-integration. For a given y, the lower limit in the x-integral is the left-most point, which is $x = y - 1$, and the upper is the right-most point, which is $x = -y^2$. Thus we have

$$\int_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} \int_{y-1}^{-y^2} \sqrt{1+x^2+y^2}dxdy.$$