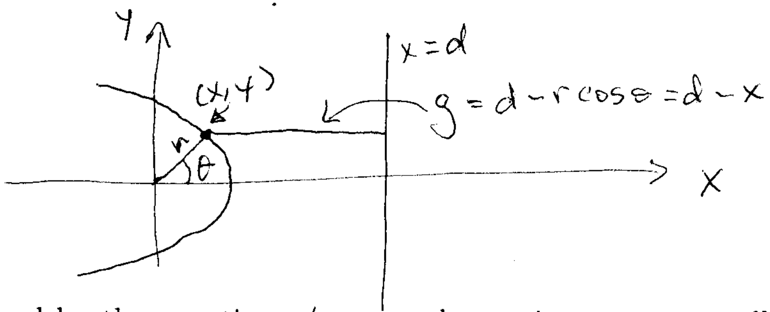
This is the general form of a conic section. Note that $g = d - r\cos\theta = d - x$.



A conic is determined by the equation $r/g = e$, where e is a constant called the eccentricity. An ellipse has eccentricity $e < 1$, a parabola has eccentricity $e = 1$, and a hyperbola has eccentricity $e > 1$. A circle is not eccentric at all ($e = 0$).

Next, we derive Kepler III. What is the equation of an ellipse in Cartesian coordinates? Start with $r/g = e$:

$$\begin{aligned}
\sqrt{x^2+y^2} &= e(d-x) \\
x^2+y^2 &= e^2(x^2-2xd+d^2) \\
x^2(1-e^2)+2de^2x+y^2 &= e^2d^2 \\
(1-e^2)[x^2+\frac{2de^2}{1-e^2}x]+y^2 &= e^2d^2 \\
(1-e^2)[x+\frac{de^2}{1-e^2}]^2+y^2 &= e^2d^2+\frac{e^4d^2}{1-e^2} = \frac{e^2d^2}{1-e^2} \\
\frac{[x+de^2/(1-e^2)]^2}{[ed/(1-e^2)]^2}+\frac{y^2}{[ed/\sqrt{1-e^2}]^2} &= 1 \\
\frac{[x+de^2/(1-e^2)]^2}{a^2}+\frac{y^2}{b^2} &= 1,
\end{aligned}$$

where $a = ed/(1-e^2)$ and $b = ed/\sqrt{1-e^2}$. This is the familiar equation for an ellipse in Cartesian coordinates. Notice that

$$b = \sqrt{a}\sqrt{ed} = \sqrt{a}\frac{2h}{c}.$$

We can compute the area of this ellipse two ways: it's πab, but it's also the integral of A' over a period T. If $A' = h$, then we get

$$\pi ab = \int_0^T A'dt = \int_0^T hdt = hT.$$

We rearrange this to read

$$T = \frac{\pi ab}{h} = \frac{\pi a\sqrt{a}}{h}\frac{2h}{\sqrt{c}} = \frac{\pi}{\sqrt{c}}a^{3/2}.$$

This is Kepler's third law.

Alright, we've shown that centripetal force with $f(r) = \frac{c}{r^2}$ implies all of Kepler's laws hold. Are there any other centripetal force laws which are also consistent with Kepler's laws? The answer is no. You can reverse all the steps we took, starting with

$$r = \frac{ed}{1+e\cos\theta}, \qquad \frac{dA}{dt} = h$$

(respectively, the planets trace out ellipses and sweep out equal areas in equal times). Define

$$\lambda = \frac{2h}{d} \qquad c = \frac{2h\lambda}{e}$$