

Lemma 3 1. $\frac{dA}{dt} = \frac{r^2}{2} \frac{d\theta}{dt}$, and so $\theta' = \frac{2h}{r^2}$

2. $\mathbf{r}' = r'\hat{l} + r\hat{l}'$

3. $\hat{l}' = \theta'\hat{m}$

4. $\hat{m}' = -\theta'\hat{l}$

Proof: In polar coordinates, the area element is given by $dA = (r^2/2)d\theta$, which gives $h = A' = (r^2/2)\theta'$. Rearranging this last equation, we get $\theta' = (2h)/r^2$, proving (1). The second item, $\mathbf{r}' = r'\hat{l} + r\hat{l}'$, follows from differentiating $\mathbf{r} = r\hat{l}$. The last two follow from the formulas for \hat{l} and \hat{m} and the chain rule. \square

5 Elliptical orbits and Kepler III

First we show that if gravity is a centripetal force whose magnitude is $f(r) = c/r^2$, then the planets trace out ellipses and Kepler III holds. Then we show the converse: if all three of Kepler's laws hold then gravity must be a centripetal force whose magnitude is $f(r) = c/r^2$.

We start with a centripetal force, so

$$\mathbf{r}'' = -f\hat{l} = -\frac{f}{r}\mathbf{r}.$$

A priori (or, before we deduce anything), f can be a function of anything. Alright, now take some cross products:

$$\mathbf{r}'' \times \hat{k} = -f\hat{l} \times \hat{k} = f\hat{m} = \frac{f}{\theta'}\hat{l}' = \frac{fr^2}{2h}\hat{l}'.$$

Suppose further that the force is an inverse square law: $f = f(r) = c/r^2$. Then

$$\mathbf{r}'' \times \hat{k} = \frac{c}{r^2} \frac{r^2}{2h} \hat{l}' = \frac{c}{2h} \hat{l}'.$$

Integrate this last equation with respect to t and recall that \hat{k} is constant:

$$\mathbf{r}' \times \hat{k} = \frac{c}{2h} \hat{l} + \mathbf{E},$$

where \mathbf{E} is a constant of integration, which must be perpendicular to \hat{k} (because everything else in the equation is). Now we have

$$\frac{c}{2h} \hat{l} + \mathbf{E} = \mathbf{r}' \times \hat{k} = (r'\hat{l} + r\hat{l}') \times \hat{k} = (r'\hat{l} + r\theta'\hat{m}) \times \hat{k} = -r'\hat{m} + r\theta'\hat{l}.$$

Take the dot product of both sides of the equation $c/(2h)\hat{l} + \mathbf{E} = -r'\hat{m} + r\theta'\hat{l}$ with \hat{l} :

$$\frac{c}{2h} = \mathbf{E} \cdot \hat{l} = r\theta' = \frac{2h}{r},$$

which we can rearrange to read

$$r = \frac{(4h^2/c)}{1 + \frac{2h}{c} \mathbf{E} \cdot (\cos \theta, \sin \theta, 0)}.$$

Now, we may as well take $\mathbf{E} = (\lambda \cos \theta, 0, 0)$ (after tilting our heads by the correct angle), so we really have

$$r = \frac{4h^2/c}{1 + (2h\lambda/c) \cos \theta} = \frac{ed}{1 + e \cos \theta}, \tag{3}$$

where $e = (2h\lambda)/c$ and $d = (2h)/\lambda$. This is the polar equation for an ellipse with eccentricity e ! (See the figure below.)