(We've also used that $\mathbf{r}$ and $\mathbf{r}'$ lie in a fixed plane, so their cross product has to be perpendicular to this plane.) Taking a derivative in time, we get

$$\mathbf{0} = \frac{d}{dt}(2h\mathbf{n}) = \frac{d}{dt}(\mathbf{r} \times \mathbf{r}')\mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{r} \times \mathbf{r}''.$$

This last equation, $\mathbf{r} \times \mathbf{r}'' = 0$, says the acceleration is parallel to the position, i.e. the motion is centripetal. □

The next lemma proves the converse:

Lemma 2 *If the motion is centripetal, then the planet orbits in a fixed plane containing the sun Kepler II holds.*

Proof: Basically, we will run the argument of the last lemma in reverse. if $\mathbf{r}$ and $\mathbf{r}''$ are parallel, then

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{r}'') = \mathbf{r} \times \mathbf{r}'' = 0,$$

and so

$$\mathbf{r} \times \mathbf{r}' = \mathbf{N},$$

where $\mathbf{N}$ is a constant vector. In particular, $\mathbf{r} \cdot \mathbf{N} = 0$, so $\mathbf{r}$ lies in the plane through the origin which is perpendicular to $\mathbf{N}$. Now write $\mathbf{N} = 2h\mathbf{n}$, where $\mathbf{n}$ has length 1 and $h > 0$ is a constant. This is our familiar $\mathbf{r} \times \mathbf{r}' = 2h\mathbf{n}$, which then implies (using equation (2))

$$\frac{dA}{dt} = h.$$

□

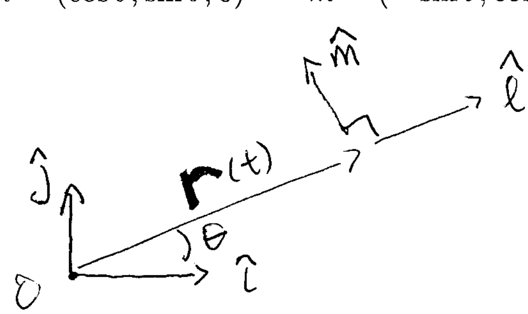
4 Moving frames

It will be convenient to use a new "moving" frame of reference, which is adapted to our moving planet. First, we may as well fix the plane of motion for the planet to be the $x - y$ plane, with $\mathbf{n} = \hat{k}$. Then

$$\mathbf{r}(t) = (x(t), y(t), 0) = (r\cos\theta, r\sin\theta, 0),$$

where we've used polar coordinates in the last equality. Remember that r and θ are functions of t! Define

$$\hat{l} = (\cos\theta, \sin\theta, 0) \qquad \hat{m} = (-\sin\theta, \cos\theta, 0).$$



Notice that we have

$$\mathbf{r}(t) = r(t)\hat{l}(t), \qquad \hat{l} \times \hat{m} = \hat{k}.$$