

(b) The only centripetal acceleration consistent with Kepler I has magnitude c/r^2 (i.e. an inverse square law).

(c) Kepler III holds if and only if the constant c is (b) is the same for each planet.

We can reproduce Newton's arguments using the tools we've developed so far in class. Beyond the historical importance, it's fascinating to see the ideas we've discussed in class used to derive equations which govern the fundamental behavior of our solar system.

2 Notation

We'll treat the planet and the sun as point particles, with their masses concentrated at their centers. Fixing our attention on one planet, we denote its position at time t by $\mathbf{r}(t)$. Its velocity and acceleration are, respectively

$$\mathbf{v}(t) = \mathbf{r}'(t) \quad \mathbf{a}(t) = \mathbf{r}''(t).$$

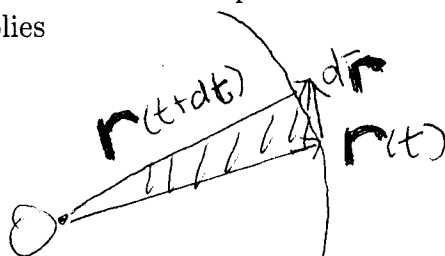
A force \mathbf{F} is called *centripetal* if \mathbf{F} and \mathbf{r} are parallel; recall this is equivalent to $\mathbf{r} \times \mathbf{F} = \mathbf{0}$. As a final bit of notation, we will denote the magnitude of a vector by writing the same letter in regular typeface. For instance, $r = |\mathbf{r}|$ and $v = |\mathbf{v}|$.

3 Centripetal forces and equal areas

The following two lemmas show that Kepler II is equivalent to the force being a centripetal force. The first lemma

Lemma 1 *If the orbit of a planet lies in a plane containing the sun and Kepler II holds, then the acceleration is centripetal.*

Proof: Recall that the magnitude of the cross product of two vectors gives the area of the parallelogram they span, which implies



$$dA = \frac{1}{2} |\mathbf{r} \times d\mathbf{r}| = \frac{1}{2} |\mathbf{r} \times \mathbf{r}' dt|.$$

(Why is this all true? The first equality is the geometric meaning of the cross product as an area and linear approximation. The second is the chain rule.) This gives us

$$\frac{dA}{dt} = \frac{1}{2} |\mathbf{r} \times \mathbf{r}'|. \quad (2)$$

Now suppose the orbit of a planet lies in a plane with unit normal \mathbf{n} . Then \mathbf{r} and *all its derivatives* lie in this plane. Now, Kepler II says $dA/dt = h$ for some constant h . Plugging this into equation (2), we see

$$\mathbf{r} \times \mathbf{r}' = 2h\mathbf{n}.$$