

Solutions to the Midterm Exam
Math 210
October 31, 2005

1. Consider the function

$$f(x, y) = \int_{1-y}^{x^2} \frac{dt}{1+t^2}.$$

(a) (4 points) Compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$.

Use the fundamental theorem of calculus and the chain rule.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{1-y}^{x^2} \frac{dt}{1+t^2} = \frac{1}{1+(x^2)^2} \cdot 2x = \frac{2x}{1+x^4}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \int_{1-y}^{x^2} \frac{dt}{1+t^2} = -\frac{1}{1+(1-y)^2} \cdot (-1) = \frac{1}{1+(1-y)^2}$$

(b) (4 points) Does f have any critical points? Be sure to explain your answer.

No, $\partial f/\partial y$ is never zero.

2. Consider the function

$$f(x, y) = x^2 + y^3 - 6xy.$$

(a) (3 points) Verify that the only critical points of f are $(0, 0)$ and $(18, 6)$.

First we compute the gradient:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x - 6y, 3y^2 - 6x).$$

Critical points occur when $\nabla f = (0, 0)$. Setting $\partial f/\partial x = 0$, we get $x = 3y$. Now plug this into $\partial f/\partial y = 0$:

$$0 = \frac{\partial f}{\partial y} = 3y^2 - 6x = 3y^2 - 18y \Leftrightarrow y = 0, 6.$$

If $y = 0$ the $x = 0$, giving us the critical point $(0, 0)$. If $y = 6$ then $x = 18$, giving us the critical point $(18, 6)$.

(b) (2 points) Classify each of these two critical points as local minima, local maxima, or saddle points.

First find the second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = -6, \quad \frac{\partial^2 f}{\partial y^2} = 6y.$$

In the case of $(0, 0)$, the discriminant is -36 , so $(0, 0)$ is a saddle point. In the case of $(18, 6)$, the discriminant is 36 and $\partial^2 f/\partial x^2 = 2$, so $(18, 6)$ is a local minimum.

(c) (3 points) Observe that $f(1, 1) = -4$. Find the equation of the tangent line to the level set $\{f = -4\}$ at the point $(1, 1)$.

Recall that the gradient is perpendicular to level sets. We have

$$\nabla f(1, 1) = (2x - 6y, 3y^2 - 6x)|_{(1,1)} = (-4, -3),$$

so the line in the perpendicular direction has slope $-4/3$. Thus the equation of the tangent line is

$$(y - 1) = -\frac{4}{3}(x - 1) \Leftrightarrow y = -\frac{4}{3}x + \frac{7}{3}.$$

(d) (2 points) Compute $D_{\vec{u}}f(1, 1)$, where $\vec{u} = (\sqrt{3}/2, 1/2)$.

$$D_{\vec{u}}f(1, 1) = \nabla f(1, 1) \cdot \vec{u} = (-4, -3) \cdot (\sqrt{3}/2, 1/2) = -\frac{4\sqrt{3} + 3}{2}.$$

3. Let $F = F(u, v)$, while $u = u(t)$ and $v = v(t)$. Call the resulting composition $f(t) = F(u(t), v(t))$.

(a) (2 points) If $u(0) = 1$, $u'(0) = 2$, $v(0) = -1$, $v'(0) = 3$ and $\nabla F(1, -1) = (8, -1)$, then what is $f'(0)$?

$$f'(0) = \frac{\partial F}{\partial u} \Big|_{(u(0), v(0))} u'(0) + \frac{\partial F}{\partial v} \Big|_{(u(0), v(0))} v'(0) = 8 \cdot 2 + (-1) \cdot 3 = 13.$$

(b) (3 points) If $u'(1) = 0$ and $v'(1) = 0$ then is it always true that $f'(1) = 0$? Be sure to explain your answer.
Yes. By the chain rule, $f'(1) = \nabla F \cdot (u'(1), v'(1)) = \nabla F \cdot (0, 0) = 0 + 0 = 0$.

(c) (3 points) If $f'(-1) = 0$, is it always true that $u'(-1) = 0 = v'(-1)$? Be sure to explain your answer.

No. It is possible that $\nabla F(u(-1), v(-1)) = (0, 0)$, which would mean that $f'(-1) = 0$. Or you could get cancellation: a positive term and a negative term adding up to give you zero.

4. (8 points) Find the maximum and minimum of $f(x, y) = xy$ subject to the constraint $x^4 + y^4 = 1$.

Use Lagrange multipliers: $\nabla f = \lambda \nabla g$ for some nonzero constant λ :

$$y = 4\lambda x^3, \quad x = 4\lambda y^3, \quad x^4 + y^4 = 1.$$

Notice that if $x = 0$ then $y = 0$ (and vice versa), which does not satisfy the constraint $x^4 + y^4 = 1$. Solve for λ in the first two equations:

$$\frac{y}{4x^3} = \lambda = \frac{x}{4y^3} \Leftrightarrow x^4 = y^4 \Leftrightarrow x = \pm y.$$

Plug this into the constraint:

$$1 = x^4 + y^4 = 2x^4 \Leftrightarrow x = \pm 2^{-1/4}, y = \pm 2^{-1/4},$$

which gives four points (all possible sign combinations). Plugging these points in to f , we see that $(2^{-1/4}, 2^{-1/4})$ and $(-2^{-1/4}, -2^{-1/4})$ are a maximum points ($f = 2^{-1/2}$) while $(2^{-1/4}, -2^{-1/4})$ and $(-2^{-1/4}, 2^{-1/4})$ are minimum points ($f = -2^{-1/2}$).

5. (a) (4 points) Evaluate $\int \int_D [x^2 y] dA$ where $D = [1, 2] \times [2, 3]$.

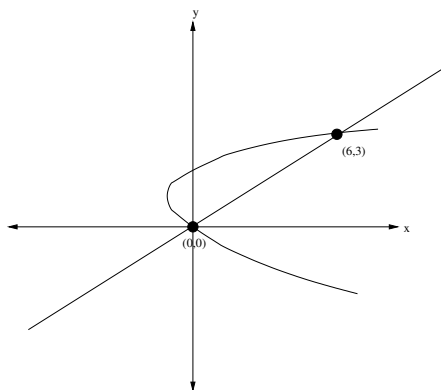
$$\begin{aligned} \int \int_D x^2 y dA &= \int_1^2 \int_2^3 x^2 y dy dx = \int_1^2 \frac{1}{2} x^2 y^2 \Big|_{y=2}^{y=3} \\ &= \frac{5}{2} \int_1^2 x^2 dx = \frac{5}{6} x^3 \Big|_1^2 = \frac{35}{6} \end{aligned}$$

(b) (4 points) Set up, but **do not** evaluate $\int \int_D e^{1-x^2-y^2} dA$, where D is bounded by $x = y^2 - y$ and $y = x/2$. (Hint: it might help to draw a picture.)

The curve $x = y^2 - y$ is a parabola opening to the right, passing through the y -axis at $y = 0$ and $y = 1$. The curve $y = x/2$ is a line through the origin with slope $1/2$. The two intersect at

$$2y = y^2 - y \Leftrightarrow 0 = y^2 - 3y = y(y - 3) \Leftrightarrow y = 0, 3.$$

Notice that the line lies to the right of the parabola in the area we're interested in ($0 \leq y \leq 3$).

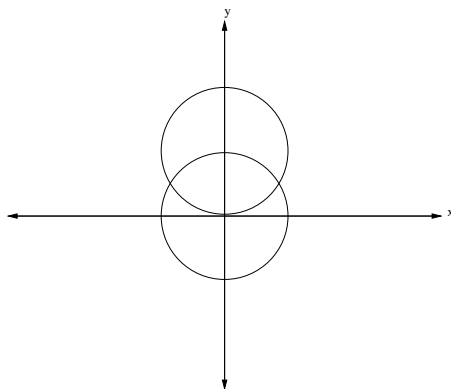


We choose horizontal slices for this integral, because for vertical slices we'd have to write down two integrals:

$$\iint_D e^{1-x^2-y^2} dA = \int_0^3 \int_{y^2-y}^{2y} e^{1-x^2-y^2} dx dy.$$

6. Consider the two circle given in polar coordinates coordinates by $\{r = 2\}$ and $\{r = 4 \sin \theta\}$.

(a) (4 points) Sketch these two circles and find their points of intersection.



Both circles have radius 2. In cartesian coordinates, the center of the circle $\{r = 2\}$ is $(0,0)$, while the center of the circle $\{r = 4 \sin \theta\}$ is $(0,2)$. The two circles intersect where

$$2 = 4 \sin \theta \Leftrightarrow \sin \theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

(And, of course, $r = 2$.) In cartesian coordinates, this gives the points $(\pm\sqrt{3},1)$.

(b) (4 points) Find the area lying inside the circle $\{r = 4 \sin \theta\}$ but outside the circle $\{r = 2\}$. (Hint: the area is given by a double integral. You might find some of the trigonometric identities on the front useful.)

We write the integral $\iint_D dA$ in polar coordinates:

$$\begin{aligned} \iint_D dA &= \int_{\pi/6}^{5\pi/6} \int_2^{4 \sin \theta} r dr d\theta = \frac{1}{2} \int_{\pi/6}^{5\pi/6} r^2 \Big|_{r=2}^{r=4 \sin \theta} d\theta \\ &= \int_{\pi/6}^{5\pi/6} 8 \sin^2 \theta - 2 d\theta = \int_{\pi/6}^{5\pi/6} 4(1 - \cos(2\theta)) - 2 d\theta = \int_{\pi/6}^{5\pi/6} 2 - 4 \cos(2\theta) d\theta \\ &= \frac{4\pi}{3} - 2 \sin(2\theta) \Big|_{\pi/6}^{5\pi/6} = \frac{4\pi}{3} - 2\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} + 2\sqrt{3}. \end{aligned}$$