Solutions to the First Midterm Exam

Math 210
Sept. 29, 2005

1. Consider the two vectors
$$\vec{v} = (3, -2, 1) \qquad \vec{u} = (2, 1, 4).$$

   (a) (4 points) Compute $\vec{u} \times \vec{v}$.
$$\vec{u} \times \vec{v} = (2, 1, 4) \times (3, -2, 1) = (1 - (-8), 12 - 2, -4 - 3) = (9, 10, -7)$$

   (b) (4 points) Find a vector $\vec{w}$ which is perpendicular to $\vec{v}$, and explain why the answer you give is correct.
   There are many possible choices. You could choose $\vec{w} = \vec{u} \times \vec{v} = (9, 10, -7)$, which you just computed. Or you could choose something to make the dot product zero, like $\vec{w} = (2, 3, 0)$ or $\vec{w} = (0, 1, 2)$. All these answers are equally correct.

2. Consider the planes Π_1 and Π_2 , where Π_1 contains the point $p_1 = (3, 2, 1)$ and has the normal vector $\vec{n_1} = (2, 0, 1)$, while Π_2 is given by the equation $x + y - z = 3$.

   (a) (4 points) Find the cosine of the angle θ between Π_1 and Π_2.
   The angle θ between the planes is the same as the angle between the two normals $\vec{n_1}$ and $\vec{n_2}$. The cosine is given by the dot product:
$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{(2, 0, 1) \cdot (1, 1, -1)}{\sqrt{4 + 0 + 1}\sqrt{1 + 1 + 1}} = \frac{1}{\sqrt{15}}.$$

   (b) (3 points) Write down the equation for Π_1.
   The equation of the plane is given by
$$\vec{n_1} \cdot (x, y, z) = \vec{n_1} \cdot (3, 2, 1) \Rightarrow 2x + z = 7.$$

   (c) (4 points) Find a vector $\vec{v}$ which is parallel to both Π_1 and Π_2.
   We want $\vec{v}$ to be parallel to both Π_1 and Π_2, so we must have $\vec{v}$ perpendicular to both $\vec{n_1}$ and $\vec{n_2}$. One such vector is the cross product:
$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = (2, 0, 1) \times (1, 1, -1) = (-1, 3, 2).$$

   (d) (4 points) Parameterize the line l of intersection between Π_1 and Π_2.
   To parameterize a line, we need to find two things: a direction vector and a basepoint. We just found the direction vector $\vec{v} = (-1, 3, 2)$ is the previous part (it's parallel to both planes), so all that remains is to find a basepoint, which is a simultaneous solution to the two equations
$$2x + z = 7 \qquad x + y - z = 3.$$
   Notice y doesn't appear in the first equation, so let's try setting $y = 0$. Then we get
$$2x + z = 7, \qquad x - z = 3.$$
   Adding these two equations together, we get $3x = 10$, or $x = 10/3$. Plugging this back into either equation, we get $z = 1/3$. So a basepoint is $(10/3, 0, 1/3)$ and a parameterization of the line of intersection is
$$l(t) = (10/3, 0, 1/3) + t(-1, 3, 2).$$

3. Consider the parameterized curve
$$\vec{r}(t) = (e^t, t, e^{-t}).$$

   (a) (4 points) Find the velocity vector of $\vec{r}$.
   Take a derivative:
$$\frac{d\vec{r}}{dt} = (\frac{d(e^t)}{dt}, \frac{d(t)}{dt}, \frac{d(e^{-t})}{dt}) = (e^t, 1, -e^{-t}).$$

(b) (3 points) Is the tangent line to $\vec{r}$ ever parallel to the yz–plane? Be sure to explain your answer.
No. The normal to the yz–plane is the vector $\vec{n} = (1, 0, 0)$, so we'd need

$$0 = \vec{n} \cdot \frac{d\vec{r}}{dt} = (1, 0, 0) \cdot (e^t, 1, -e^{-t}) = e^t.$$

However, exponentials are never zero, so this doesn't happen.

(c) (4 points) Set up, but do not evaluate, the integral to compute to arclength of the section of $\vec{r}$ for $0 \leq t \leq 1$.
The length is given by the integral of the speed:

$$L = \int_0^1 |\frac{\vec{r}}{dt}|dt = \int_0^1 |(e^t, 1, -e^{-t})|dt = \int_0^1 \sqrt{e^{2t} + 1 + e^{-2t}}dt.$$

4. Consider the function $f(x, y) = e^{x^2+y^2-1}$.

(a) (4 points) Sketch the $\{f = 1\}$ level set.
We rewrite the level set equation:

$$\{f = 1\} = \{e^{x^2+y^2-1} = 1\} = \{x^2 + y^2 - 1 = 0\} = \{x^2 + y^2 = 1\}.$$

This is the unit circle centered at the origin.

(b) (4 points) For which values of z does the level set $\{f = z\}$ not contain any points? Be sure to explain your answer.
Notice that exponentials are monotone functions: if you increase the thing in the exponent then you increase the exponential. So f is bounded from below by a lower bound for the exponent, which is $x^2 + y^2 - 1 \geq -1$. Therefore,

$$f(x, y) = e^{x^2+y^2-1} \geq e^{-1}.$$

This tells us that if we choose $z < e^{-1}$ (such as $z = 0$, or $z = 1/10$, or $z = -1$), then the level set $\{f = z\}$ will not contain any points.

5. Consider the function $f(x, y) = xe^y + \cos(xy)$.

(a) (4 points) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xe^y + \cos(xy)) = e^y - y\sin(xy), \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xe^y + \cos(xy)) = xe^y - x\sin(xy)$$

(b) (4 points) Recall that $(1, 0, \partial f/\partial x)$ and $(0, 1, \partial f/\partial y)$ are two tangent directions for the tangent plane to the graph of f at each of its points. Is this tangent plane ever parallel to the plane $x = y$? Be sure to explain your answer.
We have two tangent vectorsfor the graph, so we can compute a normal vector for the graph:

$$\vec{n}_1 = (1, 0, \frac{\partial f}{\partial x}) \times (0, 1, \frac{\partial f}{\partial y}) = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1).$$

Next, we observe that the normal vector to the plane $x = y$ is $\vec{n}_2 = (1, -1, 0)$. We can never have $\vec{n_1} \| \vec{n}_2$, because $\vec{n}_1$ has a nonzero third component, while the $\vec{n}_2$ has 0 for its third component.