Solutions to the Practice Problems

Math 115

Dec. 6, 2004

1. Find the domain of definition of each of the following functions.

   (a) $f(x) = \frac{x-2}{x^2-9}$
   f is well-defined so long as the denominator is not zero. Thus we require $x^2 - 9 \neq 0$, or $x \neq \pm 3$.

   (b) $f(x) = \sqrt{1-x^2}$
   f is well-defined so long as the thing inside the square root is greater than or equal to zero. So we have $1 - x^2 \geq 0$, or $-1 \leq x \leq 1$.

   (c) $f(x) = \ln(x^2 - 4)$
   This time we need $x^2 - 4 > 0$, which is equivalent to $|x| > 2$. Thus the domain is $(\infty, -2) \cup (2, \infty)$.

   (d) $\tan(2x)$
   We need $2x \neq (2n-1)\pi/2$, for $n = 1, 2, 3, 4, \ldots$, or $x \neq (2n-1)\pi/4$.

2. Solve each equality (or inequality) below for x.

   (a) $|x - 3| = 2$
   We have to take $x - 3 = 2$ and $-(x-3) = 2$. In the first case we have $x = 5$, and in the second case we have $x = 1$.

   (b) $\frac{x^2-1}{x-2} = 4$
   Cross-multiply to get rid of the fraction: $x^2 - 1 = 4(x-2)$. This is a quadratic equation, we we rewrite as $0 = x^2 - 4x + 7$. By the quadratic equation, there are no real roots.

   (c) $\ln(x^2 - 2x) = 0$
   Take the exponential of both sides: $x^2 - 2x = e^0 = 1$, which we rewrite as $0 = x^2 - 2x - 1$. By the quadratic equation, this has roots $x = 1 \pm \sqrt{2}$.

   (d) $|x - 4| \leq 3$
   We can rewrite this as $-3 \leq x - 4 \leq 3$. Adding 4 to both sides, we get $1 \leq x \leq 7$.

3. Evaluate each of the following limits. If the limit does not exist, be sure to explain why.

   (a) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

   $$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2$$

   (b) $\lim_{x\to 0} \sin(1/x)$
   This limit does not exist. To see this, first evaluate $f(x) = \sin(1/x)$ at $x = 1/(n\pi)$. Then $f(1/(n\pi)) = \sin(n\pi) = 0$. However, $f(2/((2n-1)\pi)) = \sin((2n-1)\pi/2) = \pm 1$. Thus we have one sequence of points $x_n = 1/(n\pi) \to 0$ where the function is zero, while for another sequence of points $\hat{x}_n = 2/((2n-1)\pi) \to 0$ where the function is either 1 pr -1. Thus the limit cannot exist.

   (c) $\lim_{x\to 0} x\sin(1/x)$
   Observe that $|\sin(1/x)| \leq 1$, so $-x \leq x\sin(1/x) \leq x$. Also,

   $$0 = \lim_{x\to 0} x, \qquad 0 = \lim_{x\to 0}(-x).$$

   Thus, by the sandwich theorem,

   $$\lim_{x\to 0} x\sin(1/x) = 0.$$

   (d) $\lim_{x\to\infty} \frac{x^3+2x^2-x+1}{2x^3-x^2+2}$

   $$\lim_{x\to\infty} \frac{x^3+2x^2-x+1}{2x^3-x^2+2} = \lim_{x\to\infty} \frac{x^{-3}(x^3+2x^2-x+1)}{x^{-3}(2x^3-x^2+2)} = \lim_{x\to\infty} \frac{1+2x^{-1}-x^{-2}+x^{-3}}{2-x^{-1}+2x^{-3}} = \frac{1}{2}.$$

   (e) $\lim_{x\to\infty} \cos(x)$
   This limit doesn't exist. Indeed, one can find a sequence $x_n = 2n\pi \to \infty$ where $\cos(x_n) = 1$, while there is also the sequence $\hat{x}_n = (2n-1)\pi \to \infty$ where $\cos(\hat{x}_n) = -1$.

4. Provide a $\delta - \epsilon$ proof for each of the following limits.

   (a) $\lim_{x\to 2}(3x-4) = 2$
   Given an $\epsilon > 0$, we wish to find a $\delta > 0$ (depending on ϵ so that if $0 < |x-2| < \delta$ then $|(3x-4)-2| < \epsilon$. Let's unravel the second inequality:

$$|(3x-4)-2| < \epsilon \Leftrightarrow -\epsilon < (3x-4)-2 < \epsilon \Leftrightarrow 6-\epsilon < 3x < 6+\epsilon \Leftrightarrow 2-\epsilon/3 < x < 2+\epsilon/3.$$

   This last statement is another way of saying $|x-2| < \epsilon/3$. So we can choose $\delta = \epsilon/3$ (or anything smaller).

   (b) $\lim_{x\to 1}(x^2+1) = 2$
   Given $\epsilon > 0$ we wish to find $\delta > 0$ (depending on ϵ) so that if $0 < |x-1| < \delta$ then $|(x^2+1)-2| < \epsilon$. Let's unravel the second inequality:

$$|(x^2+1)-2| < \epsilon \Leftrightarrow -\epsilon < (x^2+1)-2 < \epsilon \Leftrightarrow 1-\epsilon < x^2 < 1+\epsilon \Leftrightarrow \sqrt{1-\epsilon} < x < \sqrt{1+\epsilon}.$$

   Now choose

$$\delta = \min[1-\sqrt{1-\epsilon}, \sqrt{1+\epsilon}-1];$$

   with this choice, if $1-\delta < x < 1+\delta \Leftrightarrow |x-1| < \delta$, then $|(x^2+1)-2| < \epsilon$, which is the conclusion we want.

   (c) $\lim_{x\to 8} x^{1/3} = 2$
   Given $\epsilon > 0$ we wish to find $\delta > 0$ (depending on ϵ) so that if $0 < |x-8| < \delta$ then $|x^{1/3}-2| < \epsilon$. Let's unravel the second inequality:

$$|x^{1/3}-2| < \epsilon \Leftrightarrow -\epsilon < x^{1/3}-2 < \epsilon \Leftrightarrow 2-\epsilon < x^{1/3} < 2+\epsilon \Leftrightarrow (2-\epsilon)^3 < x < (2+\epsilon)^3.$$

   So choose

$$\delta = \min[2-(2-\epsilon)^3, (2+\epsilon)^3-2]$$

   and argue as before.

5. For each of the given functions $f(x)$ below, decide whether it is continuous at the point listed. Be sure to justify your answer.

   (a) $f(x) = \begin{cases} x+1 & x \le 0 \\ e^x & x > 0 \end{cases}$, $x_0 = 0$
   First observe that the right-handed and left-handed limits exist:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} e^x = e^0 = 1,$$

   and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(x+1) = 1.$$

   Moreover, these one-sided limits agree, and they are equal to the value of $f(0)$. Thus f is continuous at $x_0 = 0$.

   (b) $f(x) = |x-2|$, $x_0 = 2$
   We can write this function as

$$f(x) = \begin{cases} x-2 & x \ge 2 \\ 2-x & x < 2. \end{cases}$$

   Then one can check that the right-handed and left-handed limits as $x \to 2^{\pm}$ agree:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(x-2) = 0, \qquad \lim_{x\to 2^-} f(x) = \lim_{x\to 2}(2-x) = 0.$$

   These two limit values also agree with the value of f at $x_0 = 2$, so f is continuous there.

   (c) $f(x) = \begin{cases} \frac{x^2-1}{x-1} & x \ne 1 \\ 2 & x = 1 \end{cases}$, $x_0 = 1$
   We can write this function as

$$f(x) = \begin{cases} x+1 & x \ne 1 \\ 2 & x = 1. \end{cases}$$

   We only need to check that $\lim_{x\to 1} f(x) = 2$. Indeed,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1) = 2.$$

   Thus f is continuous at $x_0 = 1$.

(d) $f(x) = \begin{cases} \frac{x^2-1}{|x-1|} & x \neq 1 \\ 2 & x = 1 \end{cases}$, $x_0 = 1$

We can write this function as

$$f(x) = \begin{cases} x+1 & x > 1 \\ -x-1 & x < 1 \\ 2 & x = 1. \end{cases}$$

This time, the right-handed and left-handed limits disagree:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(x+1) = 2, \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1}(-x-1) = -2.$$

Thus f cannot be continuous at $x_0 = 1$.

6. A friend of your claims that a function is continuous if and only if you can write it down as one formula. Do you agree or disagree?

This is false. Indeed, the first example in the previous problem provides an example of a continuous function which cannot be written down using one formula.

7. Differentiate each of the following functions.

(a) $f(x) = x \cos x$
Use the product rule: $f' = x(-\sin x) + \cos x$

(b) $f(x) = \frac{x^2-4}{x^3+1}$
$f' = [(x^2-4)(x^3+1)^{-1}]' = 2x(x^3+1)^{-1} + (x^2-4)(x^3+1)^{-2}(-1)(3x^2) = \frac{2x(x^3+1)-3x^2(x^2-4)}{(x^3+1)^2}$
You can also use the quotient rule; you'll get the same answer either way.

(c) $\sqrt{x^2+1}$
Use the chain rule: $f' = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$.

(d) $e^{\sin(x)}$
Chain rule: $f' = e^{\sin x}(\cos x)$.

(e) $\ln(1+x^2)$
Chain rule: $f' = \frac{1}{a+x^2} \cdot 2x = \frac{2x}{a+x^2}$

8. Find the tangent line to each graph/curve at the specified point.

(a) the graph $y = x\sqrt{2x+1}$, $x_0 = 0$
First take the derivative of $y(x)$: $y' = x(2x+1)^{-1/2} + (2x+1)^{1/2}$. The slope of the tangent line at $x = 0$ (and $y = y(0) = 0$) is $y'(0) = 0 + \sqrt{1} = 1$. Thus the tangent line is $y = x$.

(b) the graph $y = \tan x$, $x_0 = \pi/4$
First take the derivatire of $y(x)$: $y' = (\sin x/\cos x)' = 1/cos^2(x)$. The slope of the tangent line at $x = \pi/4$ (and $y = y(\pi/4) = 1$) is $y'(\pi/4) = \sqrt{2}$. So the tangent line is $y - 1 = \sqrt{2}(x - \pi/4)$.

(c) the curve $4 = x^2 + 4y^2$, $(x_0, y_0) = (\sqrt{3}, 1/2)$
This time we have to differentiate implicitly. We get

$$0 = (4)' = 2x + 2(2y)(y') = 2x + 8yy'.$$

Now plug in $x = \sqrt{3}, y = 1/2$ and solve for y':

$$0 = 2\sqrt{3} + 4y' \Leftrightarrow y' = -\frac{sqrt3}{2}.$$

Thus the tangent line is $y - 1/2) = -(\sqrt{3}/2)(x - \sqrt{3})$.

(d) the curve $xy = 4$, $(x_0, y_0) = (-2, -2)$
Again, we differentiate implicitly:

$$0 = (4') = y + xy'.$$

Now plug in $x = -2 = y$ and solve for y':

$$0 = -2 - 2y' \Leftrightarrow y' = -1.$$

So the tangent line is $y + 2 = -1(x + 2)$.

(e) the curve $ye^x = 1$, $(x_0, y_0) = (0, 1)$
Differentiate implicitly to get

$$0 = y'e^x + ye^x = e^x(y + y').$$

Now plug in $x = 0, y = 1$ and solve for y':

$$0 = e^0(1 + y') \Leftrightarrow y' = -1.$$

So the tangent line is $y - 1 = -x$.

9. Consider the function $f(x) = x^5 - 5x^3 + 10x$ for $-3 \le x \le 3$.
   (a) Find all the critical points of f on the interval $[-3, 3]$.
   (b) Classify these critical points as maxima, minima, or neither.
   (c) What is the maximum value of f on $[-3, 3]$? Where does f achieve its maximum?
   (d) Where in $[-3, 3]$ is f increasing?
   (e) Where in $[-3, 3]$ is f concave up?
   (f) Sketch a graph of f.

10. A friend of yours claims that all continuous functions are differentiable. Do you agree or disagree?

    No. The function $f(x) = |x|$ is continuous everywhere, but it is not differentiable at $x_0 = 0$. The left-handed and right-handed slopes disagree.

11. Suppose a photographer is covering the 100m dash at a track and field event. She positions her camera on a stand 50m from the starting line and 5m from the track. Assume the racer run 10m/s. If the photographer tracks the racersduring the race, how fast is the angle her camera makes with the track changing when the racers pass her?

    Call the angle the photographer's camera makes θ, and call the distance along the track from the racer to the photgrapher x. Then

$$\tan\theta = \frac{x}{5}.$$

    We can differentiate this implicitly with respect to time, to get

$$\frac{\theta'}{\cos^2\theta} = (\tan\theta)' = \frac{1}{5}x' \Leftrightarrow \theta' = \frac{\cos^2\theta}{5}x'.$$

    We also have $x' = dx/dt = 10$, so $\theta' = 2\cos^2\theta$. When the racer passes directly in front of the photographer, $\theta = 0$, and so $\theta' = 2\cos(0) = 2$ then.

12. Suppose a 5ft tall person is walking towards a 15ft lamp-post at a rate of 5ft/s. When he is 15 ft from the base of the lamp-post, how fast is the length of his shadow decreasing?

    Draw two nested right triangles. The vertical leg of the big triangle is the lamp-post, which is 15ft tall, and the vertical leg of the small triangle if the walker, who is 5ft tall. These two right triangles are similar. The horizontal leg of the small triangle is the walker's shadow; we call it x. The horizontal leg of the big right triangle is $x + y$, where y is the distance from the walker to the lamp-post. Then

$$\frac{x + y}{x} = \frac{15}{5} = 3 \Leftrightarrow y = 2x.$$

    Now differentiate implicitly with respect to time t and use the fact that we know the walker's speed:

$$2x' = y' = -5 \Leftrightarrow x' = -\frac{5}{2}.$$

    So the shadow is shrinking at a rate of (5/2) ft/s.

13. Evaluate the following limits.
    (a) $\lim_{x\to 0} \frac{e^x - 1}{x^2 - x}$
    Observe that as $x \to 0$ both the numerator and the denominator approach 0. So apply L'Hopital's rule:

$$\lim_{x\to 0} \frac{e^x - 1}{x^2 - x} = \lim_{x\to 0} \frac{e^x}{2x - 1} = \frac{1}{-1} = -1.$$

(b) $\lim_{x\to 1} \frac{x-1}{\ln x}$

Observe that as $x \to 0$ both the numerator and the denominator approach 0. So apply L'Hopital's rule:

$$\lim_{x\to 1} \frac{x-1}{\ln x} = \lim_{x\to 1} \frac{1}{1/x} = 1.$$

(c) $\lim_{x\to 1} \frac{\ln x}{\sqrt{x}}$

This time, as $x \to 1$, the numerator approaches 0, but the denominator approaches 1. So we cannot use L'Hopital's rule. However, because the denominator is non-zero, the ratio is a continuous function. Thus the limit is the function value, which is $\frac{0}{1} = 0$.

14. Evaluate each of the following definite integrals.

(a) $\int_1^2 (x^2 - 3x^3 + x)dx$

$\int_1^2 (x^2 - 3x^3 + x)dx = [\frac{1}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{2}x^2]\Big|_1^2 = 8/3 - 12 + 2 - 1/3 + 3/4 - 1/2 == -89/12$

(b) $\int_0^\pi \cos x dx$

$\int_0^\pi \cos x dx = \sin x|_0^\pi = 0 - 0 = 0$

(c) $\int_0^1 x\sqrt{1+x^2}dx$

Here we make the substitution $u = 1 + x^2$:

$$\int_0^1 x\sqrt{1+x^2}dx = \frac{1}{2}\int_1^2 \sqrt{u}du = 3u^{3/2}\Big|_1^2 = 3\sqrt{8} - 3.$$

(d) $\int_0^{\pi/4} \tan x dx$

Here we make the sunstitution $u = \cos x$:

$$\int_0^{\pi/4} \tan x dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}dx = \int_1^{1/\sqrt{2}} -\frac{du}{u} = \int_{1/\sqrt{2}}^1 \frac{du}{u} = \ln u|_{1/\sqrt{2}}^1 = \ln(1) - \ln(1/\sqrt{2}) = \ln(\sqrt{2}).$$

(e) $\int_0^1 xe^{x^2} dx$

Here we make the substitution $u = x^2$:

$$\int_0^1 xe^{x^2} dx = \frac{1}{2}\int_0^1 e^u du = \frac{1}{2}e^u\Big|_0^1 = \frac{1}{2}(e-1).$$

15. Write each of the areas described below as a definite integral. You do not need to evaluate the integral.

(a) the area between $y = 0$ and $y = \sin x$, for $0 \le x \le \pi$

This is $\int_0^\pi \sin x dx$. We can write the area this way b/c $\sin x > 0$ for $0 < x < \pi$.

(b) the area between $y = 0$ and $y = \cos x$ for $0 \le x \le \pi$

This is $\int_0^\pi |\cos x| dx = \int_0^{\pi/2} \cos x dx - \int_{\pi/2}^0 \cos x dx$. Here we have to be a little careful because $\cos x$ changes sign at $x = \pi/2$.

(c) the area between $y = x$ and $y = x^3$ for $0 \le x \le 1$

This is $\int_0^1 (x - x^2)dx$. We know we have the right sign because $x > x^2$ for $0 < x < 1$.

16. Explain the inequality $|\int_a^b f(x)dx| \le \int_a^b |f(x)|dx$ in terms of areas. When are the two equal?

Recall that the integral counts a signed area; when $f < 0$ that portion of the area counts negatively. Thus one can have cancelation in $\int_a^b f(x)dx$, making it small when f actually traps large regions between its graph and the x-axis. Some of these regions will lie above the axis, and some below. However, $|f|$ is always positive, so its integral will not have any cancellations. This is why $|\int_a^b f(x)dx| \le \int_a^b |f(x)|dx$. Moreover, from this discussion one can see that $|\int_a^b f(x)dx| = \int_a^b |f(x)|dx$ precisely when f does not change sign.

17. Evaluate each of the following derivatives.

Each time we will use the Fundamental Theorem of Calculus, sometimes with the chain rule.

(a) $\frac{d}{dx}\int_0^x t^2 dt$

$\frac{d}{dx}\int_0^x t^2 dt = x^2$

(b) $\frac{d}{dx}\int_1^x t^2 dt$
$\frac{d}{dx}\int_1^x t^2 dt = x^2$

(c) $\frac{d}{dx}\int_x^1 t^3 dt$
$\frac{d}{dx}\int_x^1 t^3 dt = -x^3$

(d) $\frac{d}{dx}\int_0^{x^2} \sin t dt$
$\frac{d}{dx}\int_0^{x^2} \sin t dt = 2x\sin(x^2)$

(e) $\frac{d}{dx}\int_{\sin x}^{x} e^t dt$
$\frac{d}{dx}\int_{\sin x}^{x} e^t dt = e^x - (\cos x)e^{\sin x}$

(f) $\frac{d}{dx}\int_{x^2-3e^x}^{x^2-3e^x} t^{3/2}\sin t dt$
$\frac{d}{dx}\int_{x^2-3e^x}^{x^2-3e^x} t^{3/2}\sin t dt = 0$. Indeed, the integral itself is zero for all values of x.

18. Find the volumes described below.

(a) the volume formed by rotating $y \le x^2$ about the x-axis for $0 \le x \le 1$
Here we take vertical slices and rotate them. The resulting cylinders have volume $\pi(x^2)^2\Delta x = \pi x^4 \Delta x$. So the total volume is

$$\int_0^1 \pi x^4 dx = \frac{\pi}{5}x^5\Big|_0^1 = \frac{\pi}{5}.$$

(b) the volume formed by rotating $x \le \sin y$ about the y-axis for $0 \le y \le \pi$
Here we take horizontal slices and rotate them. The resulting cylinders have volume $\pi(\sin y)^2\Delta y = \pi\sin^2 y\Delta y$. So the total volume is

$$\int_0^\pi \pi\sin^2 y dy.$$

Unfortuantely, we haven't talked about how to integrate this yet. It turns out that we can use a trig identity: $\sin^2 y = (1/2)(1-\cos(2y))$. Then

$$\int_0^\pi \pi\sin^2 y dy = \frac{\pi}{2}\int_0^\pi (1-\cos 2y)dy = \frac{1}{2}.$$

19. Find the length of the curves described below.

(a) the graph of $y = x^2 + 1$ for $0 \le x \le 1$
The length of the graph is given by integrating the arclength functional $\sqrt{1+(y')^2} = \sqrt{1+(2x)^2} = \sqrt{1+4x^2}$. so the arclength is

$$\int_0^1 \sqrt{1+4x^2}dx.$$

(b) the curve given by $x(t) = 2\cos t, y(t) = \sin t$ for $0 \le t \le \pi$
The arclength is given by integrating $\sqrt{(x')^2+(y')^2}$. In this case, we have

$$\int_0^\pi \sqrt{((2\cos t)')^2 + ((\sin t)')^2}dt = \int_0^\pi \sqrt{4\sin^2 t + \cos^2 t}dt.$$