Solutions to the Practice Problems

Math 115

Oct. 31 2004

1. Differentiate each of the following functions.

   (a) $f(x) = x\cos(x)$
   $f' = -x\sin(x) + \cos(x)$

   (b) $f(x) = \frac{e^x}{1+x^2}$
   $f' = (e^x(1+x^2)^{-1})' = e^x(1+x^2)^{-1} - e^x(1+x^2)^{-2}(2x) = \frac{e^x(1+x^2)-2x}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}$

   (c) $f(x) = x\ln x$
   $f' = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x)$

   (d) $f(x) = e^{\sin(x)}$
   $f' = e^{\sin(x)} \cdot \cos(x)$

   (e) $f(x) = \frac{\cos(x^2)}{e^x}$
   $f' = (e^{-x}\cos(x^2))' = -e^{-x}\cos(x^2) - e^{-x}\sin(x^2)(2x) = -e^{-x}(\cos(x^2) + 2x\sin(x^2))$

   (f) $f(x) = \sqrt{1+x^2}$.
   $f' = \frac{1}{2}(1+x^2)^{-1/2}(2x) = \frac{x}{\sqrt{1+x^2}}$

2. Find the tangent line to each graph/curve at the specified point.

   (a) $f(x) = xe^x$, $x_0 = 1$
   The slope of the tangent line is given by the derivative. Here $f' = e^x(x+1)$, so $f'(1) = e^0(1+1) = 2$ and the tangent line has slope 2. This line also goes through the point $(1, f(1)) = (1, e)$. Thus the equation of the line is

   $$y - e = 2(x-1).$$

   (b) $f(x) = \tan x$, $x_0 = \pi/4$
   The slope of the tangent line is the derivative, which is $f' = 1/\cos^2(x)$. Thus the tangent line has slope $f'(\pi/4) = 1/\cos^2(\pi/4) = 2$. The line also goes through $(\pi/4, f(\pi/4)) = (\pi/4, 1)$, so its equation is

   $$y - 1 = 2(x - \pi/4).$$

   (c) $f(x) = \frac{x-2}{x^2+1}$, $x_0 = -1$
   The tangent line has slope given by

   $$f' = \frac{(x-2)'(x^2+1) - (x^2+1)'(x-2)}{(x^2+1)^2} = \frac{1+4x-x^2}{(x^2+1)^2}.$$

   Thus the slope of the tangent line is $f'(-1) = -1/2$. The line also goes through $(-1, f(-1)) = (-1, -3/2)$, so it has the equation

   $$y + 3/2 = -(x+1)/2.$$

   (d) $x^2 - y^3 = 0$, $(x_0, y_0) = (1, 1)$
   We can differentiate the equation for the curve implicitly:

   $$2x - 3y\frac{dy}{dx} = 0,$$

   and so $y' = (2x)/(3y)$. Evaluating this expression when $x = 1 = y$ (the point we're interested in on the curve; by the way, you should check this point does lie on the curve), we see $y' = 2/3$. Thus the equation of the tangent line is

   $$y - 1 = \frac{2}{3}(x-1).$$

   (e) $y\cos x + x\sin y = 0$, $(x_0, y_0) = (0, \pi)$
   Again, we differentiat implicitly:

   $$\frac{dy}{dx}\cos(x) - y\sin(x) + \sin(y) + x\cos(x)\frac{dy}{dx} = 0.$$

Plugging in $x = 0, y = \pi$, we get

$$0 = \frac{dy}{dx}\cos(0) - \pi\sin(0) + \sin(\pi) + 0\cdot\cos(\pi)\frac{dy}{dx} = \frac{dy}{dx},$$

and so the tangent line ahs slope 0 (i.e. it is a horizontal line). This means the equation of the tangent line is

$$y = \pi.$$

3. For each of the functions below, compute the derivative by evaluating the limit

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

(a) $f(x) = 2x^3 - 3x^2 + 1$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{2(x+h)^3 - 3(x+h)^2 + 1 - 2x^3 + 3x^2 - 1}{h} \\ &= \lim_{h\to 0}\frac{h(6x^2-6x) + h^2(6x-3) + 2h^3}{h} = 6x^2 - 6x. \end{aligned}$$

(b) $f(x) = (x+1)^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h+1)^2 - (x+1)^2}{h} = \lim_{h\to 0}\frac{2xh + 2h + h^2}{h} = 2x + 2.$$

4. Consider the function $f(x) = x^3 - 2x^2 + 1$ for $-2 \le x \le 2$.

(a) What are the critical points of f in the interval $[-2, 2]$?
A critical point x_0 satisfies $f'(x_0) = 0$. First we find the derivative:

$$f' = (x^3 - 2x^2 + 1)' = 3x^2 - 4x.$$

This polynomial has roots $x = 0, 4/3$, which are precisely the critical points.

(b) Characterize these critical points as local maxima or local minima as well as you can.
We will use the second derivative test, so we need to compute $f'' = 6x - 4$. Evaluating $f''(0)$ we get $f''(0) = -4 < 0$, and so $x = 0$ is a local maximum. Evaluating $f''(4/3)$ we get $f''(4/3) = 4 > 0$, and so $x = 4/3$ is a local minimum.

(c) What are the inflection points of f in the interval $[-2, 2]$?
The inflection points occur when f'' changes sign. First we find the zeroes of f'': these satisfy $0 = f''(x) = 6x - 4$, which forces $x = 2/3$. Next we check that f'' actually does have different signs to the right and left of $2/3$. Indeed, $f''(0) = -4 < 0$ and $f''(1) = 2 > 0$, so f'' does change sign. Thus $x = 2/3$ is the only inflection point.

(d) What is the maximum of f on $[-2, 2]$, and all the values of x where f assumes its maximum. Do the same thing for the minimum.
We have a candidate for the maximum: $f(0) = 1$. However, we must also check the endpoints of the interval $[-2, 2]$. In this case, $f(-2) = -15 < 1$ and $f(2) = 1$. So the maimum value of f is 1, and f achieves this maximum value twice: at $x = 0$ and $x = 2$.
We do the same thing for the minimum: $f(4/3) = -5/27$. In this case, the $f(-2) = -15 < f(4/3)$, so the minimum value of f is -15, and f achieves this minimum value once, at $x = -2$.

(e) Sketch the graph of f.

5. Suppose a 6 foot tall person is walking away from an 18 foot tall lamp-post at a rate of 2 ft/s. When the person is 12 feet from the lamp-post, how fast is her shadow lengthening?

It might help to draw a picture. The person and the lamp-post form two similar right triangles. Call the distance from the person to the lamp-post x and the distance from her to the tip of her shadow y. Then the horizontal legs of these right triangles are x and y, respectively, while the vertical legs of the right triangles are 6 and 18, respectively. Because the two triangles are similar, we have

$$\frac{y}{6} = \frac{x+y}{18} \Leftrightarrow 3y = x + y \Leftrightarrow 2y - x = 0.$$

Now differentiate this relation implicitly with respect to time t:

$$0 = 2\frac{dy}{dt} - \frac{dx}{dt} \Leftrightarrow \frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt}.$$

Thus the shadow is lengthening at half the speed the person is walking, or 1 ft/s.

6. Suppose you're making an open-top box from a rectangular sheet of cardboard, where the sheet of cardboard is 50 cm by 70 cm. You're going to make this box by cutting squares of side length x from each corner and folding the flaps up. (The squares all have to have the same side length, or else the flaps won't match up.) What value of x maximizes the volume of the box?

   It might help to draw a picture of the cardboard. Anyhow, the height of the box is the side of the square you cut out: x. The width and depth of the box are $50 - x$ and $70 - x$, respectively. Note we must take $0 \leq x \leq 50$ for to get a box in the first place. Now the volume of this box is

   $$v(x) = x(50 - x)(70 - x) = x^3 - 120x^2 + 3500x.$$

   Observe that $v(0) = 0 = v(50)$, so we won't get the largest box at the endpoints of the interval $0 \leq x \leq 50$. Now we find the critical points of v:

   $$v' = 3x^2 - 240x + 3500 = 0,$$

   so

   $$x = \frac{240 \pm 10\sqrt{156}}{6} \simeq 19.18, 60.82.$$

   The second value is too large, so we hope that $x \simeq 19.18$ is a local maximum. Indeed, if we call this critical point x_0, then

   $$v''(x_0) = 6(x_0) - 240 = (240 - 10\sqrt{156}) - 240 = -10\sqrt{156} < 0,$$

   so the critical point we found is indeed a maximum. This is the only critical point inside the interval, and the value of v is larger there than at the endpoints of the interval, so we get the largest box when we choose

   $$x = \frac{240 - 10\sqrt{156}}{6} \simeq 19.18.$$

7. Suppose you're pouring water into a cylindrical tank of radius 5 m, at a rate of 10 m^3/s. When the total volume of water is 30π, how fast is the water level rising?

   If the tank is filled to height h, then the volume of water in the tank is

   $$v(h) = 25\pi h.$$

   Now differentiate this equation with respect to time t and use the (given) fact that $dv/dt = 10$. We have

   $$10 = \frac{dv}{dt} = \frac{d}{dt}(25\pi h) = 25\pi\frac{dh}{dt}.$$

   Thus

   $$\frac{dh}{dt} = \frac{10}{25\pi},$$

   independent of what the volume or height is.

8. Let f and g be differentiable functions and suppose $f - g$ is a strictly increasing function.

   (a) If $f'(x_0) = 0$, is g increasing or decreasing near x_0?
   The fact that $f - g$ is increasing implies $0 < (f - g)' = f' - g'$, which we can rearrange to read $f' > g'$. So $f'(x_0) = 0$ forces $g'(x_0) < 0$, which implies that g is decreasing near x_0.

   (b) If f is increasing near x_0, is $g'(x_0)$ positive, negative, zero, or undetermined?
   Again, we have $f' > g'$. That f is increasing near x_0 implies $f'(x_0) > 0$, but this doesn't give us a sign on $g'(x_0)$. The deriviative $g'(x_0)$ could be a positive number smaller than $f'(x_0)$, or it could be negative. So $g'(x_0)$ is undetermined.