

Solutions to the Practice Problems
Math 115
Sept. 26, 2004

1. Find the domain of definition of the following functions:

(a) $\sqrt{1-x^2}$

We can take the square root of any non-negative number, so we only require that

$$1 - x^2 \geq 0,$$

which is equivalent to $-1 \leq x \leq 1$. This interval is the domain of definition.

(b) $\frac{x-2}{x^2-4}$

We can't let the denominator be zero, which happens when

$$0 = x^2 - 4 = (x-2)(x+2),$$

or when $x = \pm 2$. So the domain of definition is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

(c) $|x-3|$

This function is well-defined for all values of x , so the domain is the whole real line.

(d) $\frac{x^2-9}{|x-3|}$

Again, we can't let the denominator be zero. This happens when

$$0 = |x-3|,$$

or when $x = 3$. So the domain is $(-\infty, 3) \cup (3, \infty)$.

(e) $\ln(x^2-1)$

The argument of a logarithm can't be negative, so we require

$$x^2 - 1 \geq 0 \Leftrightarrow -1 \leq x \leq 1.$$

Thus the domain is $[-1, 1]$.

(f) $h(x) = f \circ g(x)$, where $g(x) = \sqrt{1-x^2}$ and $f(y) = y^2$

We encountered the function g in the first part, and found that its domain is $-1 \leq x \leq 1$. Thus the domain of the composition h must be contained in this interval. The other part of the composition, $f(y) = y^2$, is defined for all values of y . Therefore the domain of h is $-1 \leq x \leq 1$.

2. Solve for x in each of the following expressions. If the expression is an inequality, find the range of x for which the inequality holds.

(a) $x = \sqrt{1-x}$

Squaring both sides of the equation we find

$$x^2 = 1 - x \Leftrightarrow 0 = x^2 + x - 1.$$

Applying the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

(b) $e^x = 1$

Take the natural log of both sides:

$$x = \ln(e^x) = \ln(1) = 0.$$

(c) $|x-1| < 1$

We transform this into a string of inequalities without absolute values:

$$-1 < x - 1 < 1 \Leftrightarrow 0 < x < 2.$$

(d) $|x + 1| < x^2$

Again, we eliminate the absolute value:

$$-x^2 < x + 1 < x^2.$$

This time we treat the string as two separate inequalities. First we have

$$x + 1 < x^2 \Leftrightarrow x^2 - x - 1 > 0.$$

The zeroes of this quadratic function are $x = (1 \pm \sqrt{5})/2$, an one can check that $x^2 - x - 1 < 0$ for $x = 0$. Thus the inequality is satisfied for $x < (1 - \sqrt{5})/2$ and for $x > (1 + \sqrt{5})/2$. The other inequality we have is

$$-x^2 < x + 1 \Leftrightarrow x^2 + x + 1 > 0.$$

This quadratic has no real roots, and it is always positive. Thus the inequality is always satisfied. In the end, the inequality $|x + 1| < x^2$ forces x to lie in $(-\infty, (1 - \sqrt{5})/2) \cup ((1 + \sqrt{5})/2, \infty)$.

(e) $\frac{1}{x+1} = x$

So long as $x \neq -1$, we can clear the denominator by multiplying by $x + 1$:

$$1 = x(x + 1) \Leftrightarrow 0 = x^2 + x - 1.$$

Thus we have the solutions

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

We must also consider the possibility that $x = -1$, but we can verify directly that $x = -1$ is not a solution. Indeed, the left hand side of the equation is undefined for $x = -1$, but the right hand side is equal to -1 .

3. Determine whether each of the limits below exists. Compute the limits that do exist. Be sure to explain your answers!

(a) $\lim_{x \rightarrow 0} \frac{x}{x^2 - x}$

$$\lim_{x \rightarrow 0} \left(\frac{x}{x^2 - x} \right) = \lim_{x \rightarrow 0} \left(\frac{1}{x - 1} \right) = \frac{1}{0 - 1} = -1.$$

The next to last step is justified because we are taking the limit of a rational function and the denominator is non-zero at $x = 0$.

(b) $\lim_{x \rightarrow 0} \tan(1/x)$

If we evaluate this function at the sequence of points $x_n = 1/(\pi n)$, for $n = 1, 2, 3, \dots$, we find

$$\tan(1/x_n) = \tan(\pi n) = 0.$$

However, if we evaluate the function at the sequence of points $x_n = 4/((4n + 1)\pi)$, with $n = 1, 2, 3, \dots$, we find

$$\tan(x_n) = \tan\left(\frac{(4n + 1)\pi}{4}\right) = 1.$$

Thus the limit does not exist.

(c) $\lim_{x \rightarrow 0} x \tan(1/x)$

This limit doesn't exist either. First we evaluate the function at $x_n = 4/((4n + 1)\pi)$, as before. This time we find

$$x_n \tan(x_n) = \frac{4}{(4n + 1)\pi} \frac{\sin((4n + 1)\pi/4)}{\cos((4n + 1)\pi/4)} = \frac{4}{(4n + 1)\pi}.$$

Then if we evaluate the function at $x_n = 4/((4n + 3)\pi)$, we find

$$x_n \tan(x_n) = \frac{4}{(4n + 3)\pi} \frac{\sin((4n + 3)\pi/4)}{\cos((4n + 3)\pi/4)} = -\frac{4}{(4n + 3)\pi}.$$

In the first case we got large positive numbers, but in the second case we got large negative numbers. Thus $x \tan(1/x)$ oscillates very rapidly as $x \rightarrow 0$, and so the limit does not exist.

(d) $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$

$$\lim_{x \rightarrow 2} \left(\frac{x-2}{x^2-4} \right) = \lim_{x \rightarrow 2} \left(\frac{x-2}{(x-2)(x+2)} \right) = \lim_{x \rightarrow 2} \left(\frac{1}{x+2} \right) = \frac{1}{2+2} = \frac{1}{4}.$$

The next to last step is justified because we are taking the limit of a rational function and the denominator is non-zero at $x = 2$.

(e) $\lim_{x \rightarrow 2} \frac{|x-2|}{x^2-4}$

(f) This time, the function is equal to $1/(x+2)$ for $x > 2$ and equal to $-1/(x+2)$ for $x < 2$. Thus the right-sided limit is

$$\lim_{x \rightarrow 2^+} \left(\frac{|x+2|}{x^2-4} \right) = \frac{1}{2+2} = \frac{1}{4},$$

while the left-sided limit is

$$\lim_{x \rightarrow 2^-} \left(\frac{|x+2|}{x^2-4} \right) = \frac{-1}{2+2} = -\frac{1}{4}.$$

The two one-sided limits don't agree, so the limit does not exist.

(g) $\lim_{x \rightarrow 0} \frac{\tan(x)}{\sin(x)}$

$$\lim_{x \rightarrow 0} \left(\frac{\tan(x)}{\sin(x)} \right) = \lim_{x \rightarrow 0} \left(\frac{1}{\cos(x)} \right) = \frac{1}{1}.$$

The next to last step is justified because $\cos(0) = 1 \neq 0$.

(h) $\lim_{x \rightarrow \infty} e^{-x}$

Note that $e^{-x} = \frac{1}{e^x}$, so for x large and positive e^{-x} is 1 divided by a large, positive number. Thus

$$\lim_{x \rightarrow \infty} e^{-x} = 0.$$

(i) $\lim_{x \rightarrow \infty} e^x$

This limit does not exist because as x gets larger, so does e^x . In fact, for $x > 0$ we have $e^x > x$, so the numbers e^N , $n = 1, 2, 3, \dots$ grow without bound.

(j) $\lim_{x \rightarrow \infty} e^{\sin(x)}$

This limit does not exist either. First plug in $x_n = n\pi$:

$$e^{\sin(\pi/n)} = e^0 = 1.$$

Then evaluate at $x_n = (4n+1)\pi/2$:

$$e^{\sin((4n+1)\pi/2)} = e^1 = e.$$

Thus the function oscillates for large, positive x and the limit does not exist.

(k) $\lim_{x \rightarrow \infty} \sin(e^{-x})$

First recall that $\lim_{x \rightarrow \infty} e^{-x} = 0$. Thus for large and positive x , we have $\sin(e^{-x})$ is close to $\sin(0) = 0$, so

$$\lim_{x \rightarrow \infty} \sin(e^{-x}) = 0.$$

(l) $\lim_{x \rightarrow \infty} \frac{x^2}{x+1}$

Divide both top and bottom by x :

$$\frac{(1/x)(x^2)}{(1/x)(x+1)} = \frac{x}{1+1/x}.$$

Thus for x large and positive, the numerator is large and positive, while the denominator is close to 1. This implies the ratio is large and positive, so the limit does not exist.

(m) $\lim_{x \rightarrow \infty} \frac{x}{x^2+1}$

Divide both top and bottom by x :

$$\frac{(1/x)(x)}{(1/x)(x^2+1)} = \frac{1}{x+1/x}.$$

Thus for x large and positive, the denominator is large and positive, while the numerator is always 1. So the ratio is very very small, and

$$\lim_{x \rightarrow \infty} \frac{x}{x^2+1} = 0.$$

(n) $\lim_{x \rightarrow \infty} \frac{x}{x+1}$

Divide both top and bottom by x :

$$\frac{(1/x)(x)}{(1/x)(x+1)} = \frac{1}{1+1/x}.$$

Thus for x large and positive, the denominator is very close to 1, while the numerator is always 1. So the ratio is close to 1, and

$$\lim_{x \rightarrow \infty} \frac{x}{x^2+1} = 1.$$

4. Provide a δ - ϵ proof which justifies each of the following limit computations.

(a) $\lim_{x \rightarrow 2} x^3 = 8$

Given $\epsilon > 0$ we want to choose $\delta > 0$ so that

$$0 < |x - 3| < \delta \Leftrightarrow |x^3 - 8| < \epsilon.$$

Let's unravel the latter inequality:

$$|x^3 - 8| \leq \epsilon \Leftrightarrow -\epsilon < x^3 - 8 < \epsilon \Leftrightarrow 8 - \epsilon < x^3 < 8 + \epsilon.$$

Taking the cube root of the last inequality, we find

$$(8 - \epsilon)^{1/3} < x < (8 + \epsilon)^{1/3}.$$

Note that for $\epsilon > 0$, we have

$$(8 - \epsilon)^{1/3} < 2 < (8 + \epsilon)^{1/3}.$$

Now we can choose

$$\delta = \min[2 - (8 - \epsilon)^{1/3}, (8 + \epsilon)^{1/3} - 2],$$

and the inequality we want will be satisfied for $2 - \delta < x < 2 + \delta$, which is another way of saying $|x - 2| < \delta$.

(b) $\lim_{x \rightarrow -2} x^2 = 3$

(c) $\lim_{x \rightarrow 1} \frac{|x-1|}{x-1}$ DNE

Observe that for $x > 1$,

$$\frac{|x-1|}{x-1} = 1.$$

However, for $x < 1$,

$$\frac{|x-1|}{x-1} = -1.$$

Choose $\epsilon = 1/2$ and consider **any** $\delta > 0$. The interval $1 - \delta < x < 1 + \delta$ will contain points both less than and greater than 1. If the limit did, and had the value b , then for any two points $x_- < 1$ and $x_+ > 1$ we'd have

$$|x_- - b| < 1/2, \quad |x_+ - b| < 1/2.$$

Adding these two inequalities, we get

$$1 > |x_- - b| + |x_+ - b| = |x_- - b| + |b - x_+| \geq |x_- - x_+|.$$

This last term is 2 , which is bigger than 1 , contradicting our inequality. Thus the limit cannot exist.

(d) $\lim_{x \rightarrow 0} \sqrt{x} = 0$

Again we unravel the inequality $|\sqrt{x} - 0| < \epsilon$ to get

$$-\epsilon < \sqrt{x} < \epsilon \Leftrightarrow -\epsilon^2 < x < \epsilon^2.$$

Thus if we choose $\delta = \epsilon^2$ (or anything smaller) and take $0 < x < \delta = \epsilon^2$, then we have

$$0 < \sqrt{x} < \sqrt{\epsilon^2} = \epsilon.$$

5. Decide whether each of the following functions is continuous at the given point. Be sure to explain your answers!

(a) $x = 1, f(x) = |x - 1|$

First observe that $|1 - 1| = 0$. When $x < 1$, $|x - 1| = 1 - x$, which is a continuous function. So

$$\lim_{x \rightarrow 1^-} |x - 1| = \lim_{x \rightarrow 1^-} (1 - x) = 0.$$

Similarly, when $x > 1$, $|x - 1| = x - 1$ and so

$$\lim_{x \rightarrow 1^+} |x - 1| = \lim_{x \rightarrow 1^+} (x - 1) = 0.$$

The two one-sided limits agree, and so the limit is 0, which is equal to $f(1)$. Therefore the function is continuous.

(b) $x = 0, f(x) = x^{1/4}$

First observe that $0^{1/4} = 0$. However, this function only has a right-sided limit as $x \rightarrow 0$. This is because you can't take the fourth root of a negative number. Thus the limit cannot exist. It is interesting to note, however, that $\lim_{x \rightarrow 0^+} f(x) = 0 = f(0)$.

(c) $x = 2, f(x) = \begin{cases} \frac{x-2}{x^2-4} & x \neq 2 \\ 1/4 & x = 2 \end{cases}$

First, $f(2) = 1/4$. Now we have to evaluate the limit:

$$\lim_{x \rightarrow 2} \left(\frac{x-2}{x^2-4} \right) = \lim_{x \rightarrow 2} \left(\frac{1}{x+2} \right) = \frac{1}{2+2} = \frac{1}{4}.$$

The next to last step is justified because the denominator is non-zero near $x = 2$. Thus the limit exists and is equal to the value of the function, so f is continuous at $x = 2$.

(d) $x = 2, f(x) = \begin{cases} \frac{|x-2|}{x^2-4} & x \neq 2 \\ 1/4 & x = 2 \end{cases}$

This time we only need to look at the limit. Indeed, for $x < 2$ we have

$$f(x) = \frac{-(x-2)}{(x-2)(x+2)} = -\frac{1}{x+2} \rightarrow -\frac{1}{4}$$

as $x \rightarrow 2^-$. However, for $x > 2$ we have

$$f(x) = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2} \rightarrow \frac{1}{4}$$

as $x \rightarrow 2^+$. Thus the right- and left-sided limits disagree and the limit does not exist. So f is not continuous at $x = 2$.

(e) $x = 0, f(x) = \begin{cases} \tan(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

We have already seen that $\lim_{x \rightarrow 0} \tan(1/x)$ does not exist. Indeed, if we evaluate this function at the sequence of points $x_n = 1/(\pi n)$, for $n = 1, 2, 3, \dots$, we find

$$\tan(1/x_n) = \tan(\pi n) = 0.$$

However, if we evaluate the function at the sequence of points $x_n = 4/((4n+1)\pi)$, with $n = 1, 2, 3, \dots$, we find

$$\tan(x_n) = \tan\left(\frac{(4n+1)\pi}{4}\right) = 1.$$

Thus the limit does not exist, and so f is not continuous at $x = 0$.

(f) $x = 0, f(x) = \begin{cases} x \tan(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

The function f is not continuous at $x = 0$ because $\lim_{x \rightarrow 0} x \tan(1/x)$ does not exist. This limit doesn't exist either. First we evaluate the function at $x_n = 4/((4n+1)\pi)$, as before. This time we find

$$x_n \tan(x_n) = \frac{4}{(4n+1)\pi} \frac{\sin((4n+1)\pi/4)}{\cos((4n+1)\pi/4)} = \frac{4}{(4n+1)\pi}.$$

Then if we evaluate the function at $x_n = 4/((4n+3)\pi)$, we find

$$x_n \tan(x_n) = \frac{4}{(4n+3)\pi} \frac{\sin((4n+3)\pi/4)}{\cos((4n+3)\pi/4)} = -\frac{4}{(4n+3)\pi}.$$

In the first case we got large positive numbers, but in the second case we got large negative numbers. Thus $x \tan(1/x)$ oscillates very rapidly as $x \rightarrow 0$, and so the limit does not exist.

6. Prove that if two functions f and g are continuous at $x = a$ then the function $f + g$ is also continuous at $x = a$. Pick an $\epsilon > 0$ and let $\lim_{x \rightarrow a} f(x) = b_1$, $\lim_{x \rightarrow a} g(x) = b_2$. Because f and g are continuous at $x = a$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Leftrightarrow |f(x) - b_1| < \epsilon/2$$

and

$$0 < |x - a| < \delta_2 \Leftrightarrow |g(x) - b_2| < \epsilon/2.$$

If we take $\delta = \min(\delta_1, \delta_2)$ and suppose $0 < |x - a| < \delta$, then

$$|(f(x) + g(x)) - (b_1 + b_2)| = |(f(x) - b_1) + (g(x) - b_2)| \leq |f(x) - b_1| + |g(x) - b_2| < \epsilon/2 + \epsilon/2 = \epsilon.$$

7. Find two functions $f(x)$ and $g(x)$ such that $f(0) = 0 = g(0)$ and

(a) $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)}$ DNE

We can choose $f(x) = x$ and $g(x) = x^2$. Then the ratio f/g is unbounded for x close to 0.

(b) $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = 0$

We can choose $f(x) = x^2$ and $g(x) = x$. Then the ratio $f/g = h(x) = x$, which is continuous, and $h(0) = 0$.

(c) $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = 2$

We can choose $f(x) = x^2 + 2x$ and $g(x) = x$. Then the ratio is $f/g = h(x) = x + 2$, which is continuous, and $h(0) = 2$.

8. Suppose $f(x)$ is a function defined for $0 \leq x \leq 1$, such that $0 \leq f(x) \leq 1$. For each of the statements below, either argue why it is correct or find a counter-example.

- (a) If f has an inverse function (i.e. there is a $g = g(y)$ such that $g(f(x)) = x$) then f is continuous.

False. Consider

$$f(x) = \begin{cases} x/2 & 0 \leq x < 1/2 \\ x/2 + 3/4 & 1/2 \leq x \leq 1. \end{cases}$$

This function is invertible (i.e. it passes the vertical line test), but it's not continuous because $\lim_{x \rightarrow 1/2} f(x)$ does not exist.

- (b) If f is continuous then it has an inverse function.

False. Consider $f(x) = x(1 - x) = x - x^2$. This is continuous and satisfies all the other hypotheses, yet it is not invertible for $0 \leq x \leq 1$. Indeed, it assumes all values between 0 and 1/4 twice.

- (c) If f is continuous and it has an inverse function then the inverse function is also continuous.

True. Recall that one obtains the graph of the inverse function by reflecting the original graph across the line $x = y$. Because f is continuous, we can draw the graph of f without lifting our pen. The same is true for a reflection of this graph, and so we can draw the graph of f^{-1} without lifting our pen. Therefore f^{-1} is continuous.