Selected Solutions for Homework 4

Math 115

Sept. 29, 2004

1. (problem 1.3.54) We have
$$f(x) = \begin{cases} x^2 & x \neq 0 \\ 4 & x = 0 \end{cases} \qquad g(x) = 2x.$$
First we compute $\lim_{x\to 0} f(g(x))$:
$$\lim_{x\to 0} f(g(x)) = \lim_{x\to 0}(2x)^2 = \lim_{x\to 0} 4x^2 = 4(0)^2 = 0.$$
Next we compute $f(\lim_{x\to 0} g(x))$:
$$f(\lim_{x\to 0} g(x)) = f(\lim_{x\to 0} 2x) = f(0) = 4.$$
These two are not equal.

2. (problem 1.4.4) Can the graph of a function cross its asymptotes? The graph can cross its horizontal and slanted asymptotes as many times as you please. Consider $f(x) = e^{-x}\cos(x)$ (horizontal asymptote at $y = 0$) and $g(x) = x + e^{-x}\cos(x)$ (slanted asymptote at $x = y$). Both these functions cross their asymptotes infinitely often, at half-integer multiples of π. However, the graph cannot cross a vertical asymptote. If it did, then the graph would fail the vertical line test for values of x close to the vertical asymptote.

3. (problem 1.4.20)
$$\lim_{x\to\infty} \frac{2x^2 - x + 1}{4x^2 - 3x - 1} = \lim_{x\to\infty} \frac{x^{-2}(2x^2 - x + 1)}{x^{-2}(4x^2 - 3x - 1)} = \lim_{x\to\infty} \frac{2 - x^{-1} + x^{-2}}{4 - 3x^{-1} - x^{-2}} = \frac{2}{4} = \frac{1}{2}.$$

4. (problem 1.4.25)
$$\lim_{x\to\infty} \frac{x^2 - \sin x}{x^2 + 4x - 1} = \lim_{x\to\infty} \frac{x^{-2}(x^2 - \sin x)}{x^{-2}(x^2 + 4x - 1)} = \lim_{x\to\infty} \frac{1 - x^{-2}\sin x}{1 + 4x^{-1} - x^{-2}} = \frac{1}{1} = 1.$$

5. (problem 1.4.51) Consider $f(x) = e^{-x}\cos x$. This function has a horizontal asymptote of $y = 0$, which it only approaches as $x \to \infty$ (i.e. in the positive direction). The graph crosses the asymptote infinitely often, at $x = \pi/2, 3\pi/2, 5\pi/2, \ldots$

6. (problem 1.5.41) Verify that $\lim_{x\to\infty} \frac{2}{x^3} = 0$.
Given $\epsilon > 0$, we want to find N such that if $x > N$ then $|2/x^3| < \epsilon$. Ok, so let's unravel the inequality:
$$|\frac{2}{x^3}| < \epsilon \Leftrightarrow -\epsilon < \frac{2}{x^3} < \epsilon \Leftrightarrow -\epsilon x^3 < 2 < \epsilon x^3.$$
Notice that x^3 has the same sign as x, and we are looking at positive value of x. For $x > 0$, we have $-\epsilon x^3 < 0 < 2$, so that inequality is satisfied. Now look at the other inequality:
$$2 < \epsilon x^3 \Leftrightarrow \frac{2}{\epsilon} < x^3 \Leftrightarrow \left(\frac{2}{x^3}\right)^{1/3}.$$
Thus we can choose $N = (2/\epsilon)^{1/3}$ (or anything larger).

7. (problem 1.5.51) Let
$$f(x) = \begin{cases} 2x & x < 1 \\ x^2 + 3 & x > 1. \end{cases}$$
We want to show that $\lim_{x\to 1} f(x) \neq 2$. This means we want to show that
$$|f(x) - 2|$$
can be large in any interval centered on $x = 1$, no matter how small the interval. In fact, we can always make this difference at least 1, so we choose $\epsilon = 1$. We will show that for any $\delta > 0$, there are points x such that $|x - 1| < \delta$ and yet $|f(x) - 2| \geq 1$. Indeed, choose $x = 1 + \delta/2$. Then
$$f(x) = f(1 + \delta/2) = (1 + \delta/2)^2 + 3 \geq 3,$$
because anything squared is non-negative. This is a crude lower bound, but it works. For this choice of $x = 1 + \delta/2$, we have
$$|f(x) - 2| \geq |3 - 2| = 1,$$
which completes the argument.