

Selected Solutions for Homework 1

Math 115

Sept. 9, 2004

Note on grading: I graded problems 0.1.21, 0.2.32, 0.4.1, 0.4.37, and 0.7.27; two points per problem. The solutions to all these problems are below, as well as solutions to some other problems I liked. In general,

1. (Problem 0.1.2) The string

$$4 < x < -4$$

claims that the number x is simultaneously less than -4 and greater than 4 . This would require, in particular, that x is both a positive number and a negative number, which is impossible.

2. (Problem 0.1.21) The inequality $|x - 3| < 4$ means

$$-4 < x - 3 < 4.$$

Adding 3 to both sides of this inequality, we see this is equivalent to

$$-1 < x < 7.$$

3. (Problem 0.2.3)

If a graph fails the vertical line test, then there is a value of x , say $x = x_0$, with at least two values of y , say $y = y_1$ and $y = y_2$ associated to it. However, if the graph is the graph of a function, then there is only one value of y associated to each x . Indeed, in this case how does one decide whether $f(x_0) = y_1$ or $f(x_0) = y_2$?

4. (Problem 0.2.5)

Are the points

$$a = (1, 2), \quad b = (3, 6), \quad c = (0, 0)$$

collinear? First note that the line through a and c has slope

$$\frac{2 - 0}{1 - 0} = 2,$$

and is given by

$$y - 0 = 2(x - 0),$$

or $y = 2x$. The line through b and c has slope

$$\frac{6 - 0}{3 - 0} = 2,$$

and is given by

$$y - 0 = 2(x - 0),$$

or $y = 2x$. So these three points are collinear.

5. (Problem 0.2.32) First recall that two lines are parallel if and only if they have the same slope. These lines are perpendicular if and only if their slopes are negative reciprocals (*i.e.* if the slopes are m_1 and m_2 then $m_1 m_2 = -1$). In this case, the two lines have slopes 2 and 4, so they are neither parallel nor perpendicular.
6. (Problem 0.4.1) Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be an odd degree polynomial (so n is odd). There are two key points to make: first, when $|x|$ is large the first term $a_n x^n$ is much larger than the sum of all the other terms. In other words, when $|x|$ is large the $a_n x^n$ term dominates everything else. Second, this term $a_n x^n$ has the opposite sign for x large and positive as it does for x large and negative. The combination of these two facts implies $p(x)$ has the opposite sign for x large and positive as it does for x large and negative. Thus $p(x)$ has a real zero. Geometrically, one can see this in that the graph of $p(x)$ crosses the x -axis at least once, because the graph has to pass from above the axis to below the axis. This crossing point is a real zero.
7. (Problem 0.4.37) We want to find all solutions to

$$x = \frac{x^2}{x^2 + 1}.$$

First cross-multiply:

$$x^2 = x(x^2 + 1) = x^3 + x.$$

Then move everything to one side of the equation and factor:

$$0 = x^3 - x^2 + x = x(x^2 - x + 1).$$

Observe that one root is $x = 0$. To find the other roots, we set

$$0 = x^2 - x + 1$$

and use the quadratic equation:

$$x = \frac{1 \pm \sqrt{1 - 4}}{2}.$$

Because the expression under the radical sign is negative, we see that there are no real roots aside from $x = 0$.

8. (Problem (0.4.42) We know each of the polynomials listed below has the prescribed number of zeroes because we have written each polynomial in its factored form, from which we can read off all the zeroes.
- (a) no zeroes: $f(x) = x^4 + 1$ (always positive)
 - (b) one zero: $f(x) = x^4$
 - (c) two zeroes: $f(x) = (x - 1)^2(x - 3)^2$ (zeroes at 1, 3)
 - (d) three zeroes: $f(x) = (x + 1)(x + 1)(x - 2)^2$ (zeroes at $-1, 1, 2$)
 - (e) four zeroes: $f(x) = (x - 1)(x + 1)(x - 2)(x + 2)$ (zeroes at $-1, 1, -2, 2$)
9. (Problem 0.7.27) The graph is shifted down by 3 units. I can't draw pictures here, so you'll have to use your imagination a little.
10. (Problem 0.7.64) We examine iterates of the function $f(x) = x^2$, first starting with $x_0 = 1/2$. The iterates are

$$x_0 = \frac{1}{2}, x_1 = f(x_0), x_2 = f(x_1), \dots$$

Indeed, if we compute the first several iterates, we see

$$x_1 = \frac{1}{4} = \frac{1}{2^2}, \quad x_2 = \frac{1}{16} = \frac{1}{4^2} = \frac{1}{2^4}, \quad x_3 = \frac{1}{256} = \frac{1}{16^2} = \frac{1}{2^8}, \dots$$

The general pattern is that

$$x_n = \frac{1}{2^{2^n}}, n = 1, 2, 3, \dots$$

The salient point is that for n large, the number $x_n = 1/2^{2^n}$ is tiny, and it gets tinier as n gets larger. So the iterates x_n approach zero.

Next we start the iterations with $x_0 = 2$. In this case, when we compute the first several iterates we find

$$x_1 = 4 = 2^2, \quad x_2 = 16 = 2^4, \quad x_3 = 256 = 2^8, \dots$$

the general pattern is that

$$x_n = 2^{2^n}.$$

This time the iterates grow without bound, so they don't approach any finite number.