Solutions to the Second Midterm Exam

Math 115

Nov. 5, 2004

1. Differentiate the following functions.

   (a) (2 points) $f(x) = \frac{\cos(x)}{1+x^2}$

   $f' = [\cos(x)(1+x^2)^{-1}]' = -\sin(x)(1+x^2)^{-1} - 2x\cos x(1+x^2)^{-2} = \frac{-(1+x^2)\sin x - 2x\cos x}{(1+x^2)^2}$

   (b) (2 points) $f(x) = e^x \sin(x)$

   $f' = (e^x \sin x)' = e^x \cos x + e^x \sin x = e^x(\cos x + \sin x)$

   (c) (2 points) $f(x) = \sin(x^2)$

   $f' = (\sin(x^2))' = 2x\cos(x^2)$

2. (8 points) Evaluate the limit of difference quotients to find the derivative of $f(x) = 3x - x^2$. You must use the definition of a derivative in terms of the limit of difference quotients, instead of the rules regarding the differentiation of functions.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h) - (x+h)^2 - 3x + x^2}{h} \\ &= \lim_{h\to 0} \frac{3x + 3h - x^2 - 2xh - h^2 - 3x + x^2}{h} = \lim_{h\to 0} \frac{3h - 2xh - h^2}{h} = 3 - 2x. \end{aligned}$$

3. Compute each of the following limits. Be sure to justify your answers.

   (a) (4 points) $\lim_{x\to 2} \left(\frac{x^2-4}{x^2-4x+4}\right)$

   Notice that as $x \to 2$, both the numerator and the denominator approach 0. So we can use L'Hopital's rule:

$$\lim_{x\to 2} \frac{x^2-4}{x^2-4x+4} = \lim_{x\to 2} \frac{(x^2-4)'}{(x^2-4x+4)'} = \lim_{x\to 2} \frac{2x}{2x-4}.$$

   This last limit does not exist because the numerator stays bounded while the denominator blows up. One can also evaluate this limit by factor the two quadratics.

   (b) (4 points) $\lim_{x\to 1} \left(\frac{\ln x}{\sqrt{x}}\right)$

   In this case, as $x \to 1$, the denominator approached 1 while the numerator approaches 0. Thus

$$\lim_{x\to 1} \frac{\ln x}{\sqrt{x}} = \frac{0}{1} = 0.$$

4. (8 points) Suppose a farmer wants to fence in a rectangular plot of land along a river (which you can assume runs in a straight line in the north-south direction). This farmer has 100 meters of fencing to use. What is the largest area he a fence in, and what are the side-lengths of the largest rectangle?

   Let x be the length of the two sides pf the rectangle perpendicular to the river. Then the side of the rectangle opposite the river has length $100 - 2x$. The farmer doesn't have to run the fence along the river. The total area is

$$A(x) = x(100 - 2x) = 100x - 2x^2,$$

   with $0 \leq x \leq 50$. We want to maximize the area function $A(x)$, so we first look for critical points.

$$0 = A'(x) = 100 - 4x \Leftrightarrow x = 25.$$

   Next we want to see if $x = 25$ is a local max or local min, so we apply the second derivative test:

$$A''(x) = -4, \qquad A''(25) = -4 < 0,$$

   so $x = 25$ is a local max. The area at $x = 25$ is

$$A(25) = 25 \cdot 50 = 1250.$$

   Finally, we check that the area is not greatest at the endpoints of the interval $[0, 50]$. Indeed, when $x = 0$ or $x = 50$, the area fenced in is 0. So the largest area he can fence is is 1250 m^3 and the plot of land is 25×50 meters.

5. Consider the function $f(x) = x^3 + x^2 - x$ for $-2 \le x \le 2$.

   (a) (3 points) Find all the critical points of f in the interval $[-2, 2]$.
   The critical points satisfy
   $$0 = f'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1),$$
   so the critical points are $x = -1, 1/3$.

   (b) (3 points) Classify these critical points as local maxima, local minima, or neither.
   We apply the second derivative test. First, note that
   $$f''(x) = 6x + 2.$$
   Now evaluate f'' at each critical point:
   $$f''(-1) = -4 < 0, \quad f''(1/3) = 4 > 0$$
   so $x = -1$ is a local maximum, while $x = 1/3$ is a local minimum.

   (c) (2 points) What is the largest possible value of f on $[-2, 2]$, and for which values of x is $f(x)$ equal to this maximal value?
   We need to compare f at the critical points and the endpoints. One can compute that $f(-2) = -2$, $f(-1) = 1$, $f(1/3) = -5/27$, $f(2) = 10$. Thus the largest value of f is 10, and f assumes this lasrgest value only when $x = 2$.

   (d) (2 points) Where in the interval $[-2, 2]$ is f increasing?
   the function f is increasing precisely when $f' > 0$. We know that the zeroes of f' are $x = -1, 1/3$, so we can test some values:
   $$f'(-3/2) = 11/4 > 0, \qquad f'(0) = -1 < 1, \qquad f'(1) = 4 > 0.$$
   Thus $f' > 0$, and f is increasing, on $[-2, -1) \cup (1/3, 2]$.

   (e) (2 points) Where is the interval $[-2, 2]$ is f concave up?
   The function f is concave up precisely when $f'' > 0$. We have
   $$0 < f'' = 6x + 2 \Leftrightarrow x > -1/3,$$
   so f is concave up on $(-1/3, 2]$.

6. (8 points) A photographer is tracking a plane which is flying at an altitude of 2 km and speed 500 km/h. Let θ denote the angle between her line of sight and the vertical direction. When the plane passes directly over her head, how fast is θ changing?

   Let x be the horizontal distance from the plane to the photpgrapher. Then there is a right triangle with legs x and 2, and angle θ at the intersection of the hypotneus and leg of length 2. Thus
   $$\tan\theta = \frac{x}{2}.$$
   We are given $dx/dt = 500$ and want to find $d\theta/dt$. So differentiate the relationship between θ and x above with respect to t:
   $$250 = \frac{1}{2}\frac{dx}{dt} = \frac{d}{dt}\tan\theta = \frac{1}{\cos^2\theta}\frac{d\theta}{dt}.$$
   We can solve for $d\theta/dt$ to get
   $$\frac{d\theta}{dt} = 250\cos^2\theta.$$
   When the plane is directly overhead, the angle θ is zero, so $\cos^2\theta = 1$. So at that time $d\theta/dt = 250$.