Solutions to the first midterm exam

1. Find the domain of definition of the following functions.

   (a) (5 points) $f(x) = \sqrt{4 - x^2}$
   The term $4 - x^2$ inside the square root must be non-negative. So we have

   $$4 - x^2 \geq 0 \Leftrightarrow 4 \geq x^2 \Leftrightarrow 2 \geq \sqrt{x^2} = |x|.$$

   Thus the domain of the function is $[-2, 2]$.

   (b) (5 points) $f(x) = \frac{x-2}{x^2-9}$
   We cannot allow the denominator to be zero, so we require

   $$0 \neq x^2 - 9 = (x + 3)(x - 3).$$

   Thus the domain is $x \neq \pm 3$, or $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

2. Compute the following limits. If the limit does not exist, state so and explain why.

   (a) (4 points) $\lim_{x\to 2} \left(\frac{x-2}{x^2-4}\right)$

   $$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4}.$$

   The next to last step is justified because $x + 2 \neq 0$ for x near 2.

   (b) (4 points) $\lim_{x\to\infty} \left(\frac{x^2}{2-x^2}\right)$

   $$\lim_{x\to\infty} \frac{x^2}{2-x^2} = \lim_{x\to\infty} \frac{x^{-2}(x^2)}{x^{-2}(2-x^2)} = \lim_{x\to\infty} \frac{1}{2x^{-2}-1} = \frac{1}{-1} = -1.$$

   (c) (4 points) $\lim_{x\to\infty} \left(e^{-\sin(x)}\right)$
   This limit does not exist. Recall that $\sin x$ oscillates between -1 and 1. Indeed, for $x = (4n+1)\pi/2$, $n = 1, 2, 3, \ldots$, we have $\sin((4n+1)pi/2) = 1$ and so $e^{-\sin(x)} = e^{-1} = 1/e$, while for $x = (4n+3)\pi/2$, $n = 1, 2, 3, \ldots$, we have $\sin((4n+3)\pi/2) = -1$ and so $e^{-\sin(x)} = e^1 = e$. Because we can find arbitrarily large values of x for which the function is either e or $1/e$, the limit does not exist.

3. You may wish to recall the formal definition of a limit: we say the function $f(x)$ approaches a limit b as x approaches a when for every $\epsilon > 0$ there exists $\delta > 0$ such that

   $$0 < |x - a| < \delta \Rightarrow |f(x) - b| < \epsilon.$$

   Consider the function $f(x) = 2x + 1$.

   (a) (5 points) Find a $\delta > 0$ such that if $0 < |x - 2| < \delta$ then $|(2x + 1) - 5| < \frac{1}{100}$.
   We want to find a δ so that for $|x - 2| < \delta$ we have

   $$|(2x + 1) - 5| < \frac{1}{100}.$$

   Unraveling this latter inequality, we find

   $$-\frac{1}{100} < (2x+1) - 5 = 2x - 4 < \frac{1}{100} \Leftrightarrow \frac{399}{100} < 2x < \frac{401}{100} \Leftrightarrow \frac{399}{200} < x < \frac{401}{200}.$$

   If we unravel $|x - 2| < \delta$, we find

   $$2 - \delta < x < 2 + \delta.$$

   Thus any choice

   $$\delta \leq \min\{2 - \frac{399}{200}, \frac{401}{200} - 2\} = \frac{1}{200}$$

   works.

(b) (5 points) Provide a formal δ-ϵ proof that $\lim_{x\to 2}(2x+1) = 5$.

Given an $\epsilon > 0$, we want to find δ so that for $|x-2| < \delta$ we have

$$|(2x+1) - 5| < \epsilon.$$

Unraveling this latter inequality, we get

$$-\epsilon < (2x+1) - 5 = 2x - 4 < \epsilon \Leftrightarrow 4 - \epsilon < 2x < 4 + \epsilon \Leftrightarrow 2 - \frac{\epsilon}{2} < x < 2 + \frac{\epsilon}{2}.$$

Thus if we choose $\delta = \epsilon/2$ (or anything smaller) and

$$|x-2| < \delta = \epsilon/2 \Leftrightarrow 2 - \epsilon/2 < x < 2 + \epsilon/2,$$

we then have $|(2x+1) - 5| < \epsilon$. This completes the proof.

4. (10 points) Consider the function

$$f(x) = \begin{cases} \frac{|x-1|}{x-1} & x \neq 1 \\ 1 & x = 1. \end{cases}$$

Is f continuous at $x = 1$? Be sure to explain your answer.

No, f is not continuous at $x = 1$. Indeed, for $x > 1$, we have $f(x) = 1$, and so

$$\lim_{x\to 1^+} f(x) = 1.$$

However, for $x < 1$, we have $f(x) = -1$, and so

$$\lim_{x\to 1^-} f(x) = -1.$$

Because the one-sided limits disagree, the limit $\lim_{x\to 1} f(x)$ does not exist. So f cannot be continuous at $x = 1$.

5. Be sure to explain your answers to these questions.

(a) (4 points) Find a bounded function $f(x)$ such that both the one-sided limits $\lim_{x\to 0^+}(f(x))$ and $\lim_{x\to 0^-}(f(x))$ exist, and yet $\lim_{x\to 0}(f(x))$ does not exist.

Let

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0. \end{cases}$$

Then

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1,$$

while

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -1 = -1.$$

So both one-sided limits exist, but they disagree. Therefore $\lim_{x\to 1} f(x)$ does not exist.

(b) (4 points) Find a bounded function $g(x)$, defined for $0 < x < 1$, such that the one-sided limit $\lim_{x\to 0^+}(g(x))$ does not exist.

Let $g(x) = \sin(1/x)$. This is a bounded functions; indeed, it takes values in the interval $[-1, 1]$. If we evaluate g at $x = 1/(n\pi)$, then we get

$$g(1/n\pi) = \sin(n\pi) = 0.$$

However, if we evaluate g at $x = 2/((4n+1)\pi)$, then we get

$$g(2/((4n+1)\pi)) = \sin((4n+1)\pi/2) = 1.$$

So g oscillates very rapidly as $x \to 0$, which means the limit cannot exist.