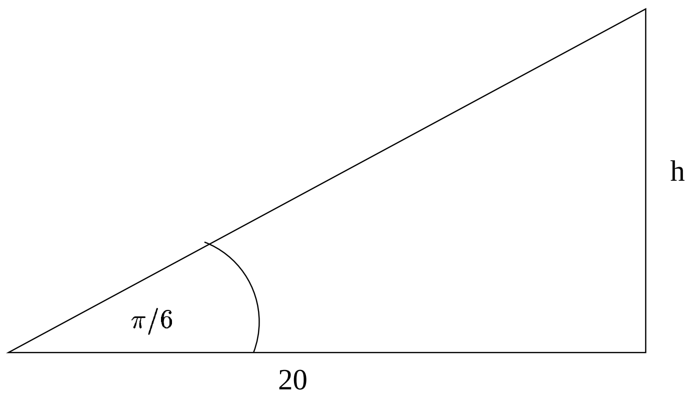
Solutions to the Practice Problems
Math 1060
December 1, 2002

1. Suppose the shadow of a tree is 20 feet long when the angle of elevation the sum makes with the ground is $\pi/6$. Find the height of the tree.

   Call the height of the tree h (see the figure below). Then we know that

$$\frac{h}{20} = \tan(\pi/6) = \frac{1}{\sqrt{3}},$$

   which we can rearrange to read $h = 20/\sqrt{3}$.

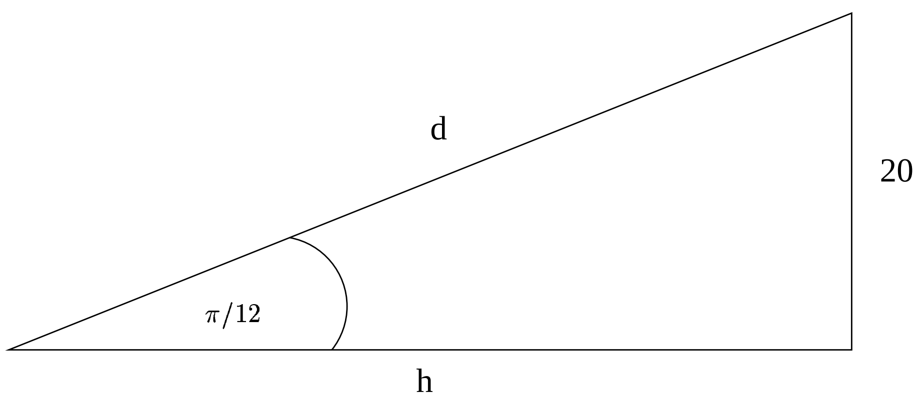


2. Suppose a plane is flying at 600 miles per hour and a height of 20 miles. A direct line from a satellite dish on the ground to the plane makes an angle of $\pi/12$ with the ground.

   (a) What is the horizontal distance from the plane to the satellite dish?

   Let h be the horizontal distance from the plane to the satellite dish (see the figure below). Then $\tan(\pi/12) = 20/h$, and so

$$h = \frac{20}{\tan(\pi/12)} = 74.64 \text{ miles.}$$



   (b) What is the distance between the plane and the satellite dish?

   Again, refer to the figure, and let d be the distance between the plane and the satellite dish. Then $\sin(\pi/12) = 20/d$, and so

$$d = \frac{20}{\sin(\pi/12)} = 77.27.$$

   (c) How much time elapses before the plane passes over the satellite dish, assuming that it flies at the same height and speed, in a straight line directly over the satellite dish?

   We want to know the amount of time which elapses before the plane travels $h = 74.64$ miles. The plane is flying at 600 miles per hour, so the amount of time is

$$t = \frac{74.64}{600} = .1244 \text{ hours,}$$

   which is 7.464 minutes.

3. If $\sin\theta = -1/\sqrt{2}$ and $\cos\theta > 0$ find

(a) $\cos\theta$
We know that $\sin^2\theta + \cos^2\theta = 1$, so

$$\cos^2\theta = 1 - \sin^2\theta = 1 - (-1/\sqrt{2})^2 = 1 - 1/2 = 1/2.$$

Using the fact that $\cos\theta > 0$, we see that $\cos\theta = \sqrt{1/2} = 1/\sqrt{2}$.

(b) $\csc\theta$

$$\csc\theta = \frac{1}{\sin\theta} = -\sqrt{2}$$

(c) $\tan\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-1/\sqrt{2}}{1/\sqrt{2}} = -1.$$

4. If $\tan\theta = 1$ and $\cos\theta < 0$ find

(a) $\cos\theta$
We know that

$$1 = \tan\theta = \frac{\sin\theta}{\cos\theta},$$

which we can rearrange as $\sin\theta = \cos\theta$. Now use this equation the identity

$$1 = \cos^2\theta + \sin^2\theta = 2\cos\theta,$$

which implies $\cos^2\theta = 1/2$. Taking the negative square root (because we're given $\cos\theta < 0$), we get $\cos\theta = -1/\sqrt{2}$.

(b) $\sin\theta$
Above we saw $\sin\theta = \cos\theta = -1/\sqrt{2}$.

(c) $\csc\theta$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-1/\sqrt{2}} = -\sqrt{2}.$$

5. If $\cot\theta = -\sqrt{3}$ and $\sin\theta < 0$ find

(a) $\sin\theta$
We know that

$$-\sqrt{3} = \cot\theta = \frac{\cos\theta}{\sin\theta},$$

which we can rearrange to read $-\sqrt{3}\sin\theta = \cos\theta$. Now we use this equation in the identity

$$1 = \cos^2\theta + \sin^2\theta = (-\sqrt{3}\sin\theta)^2 + \sin^2\theta = 3\sin^2\theta + \sin^2\theta = 4\sin^2\theta,$$

or $\sin^2\theta = 1/4$. Taking the negative square root (because $\sin\theta < 0$), we get $\sin\theta = -1/2$.

(b) $\cos\theta$
Above we found that $\cos\theta = -\sqrt{3}\sin\theta = \sqrt{3}/2$.

(c) $\sec\theta$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}.$$

6. Consider the function $F(t) = 1 + 2\cos(2(t - \pi/3))$

(a) What is the average value of F?
Recall that adding 1 translates the graph of the cosine curve up by 1, which increases the average value by 1. As the average value of a regular cosine curve is 0, the average value of F is 1.

(b) What is the amplitude of F?

Recall that the factor of 2 next to the cosine acts to expand the curve vertically by a factor of 2. As a regular cosine curve has an amplitude of 1, F has an amplitude of $2 \cdot 1 = 2$.

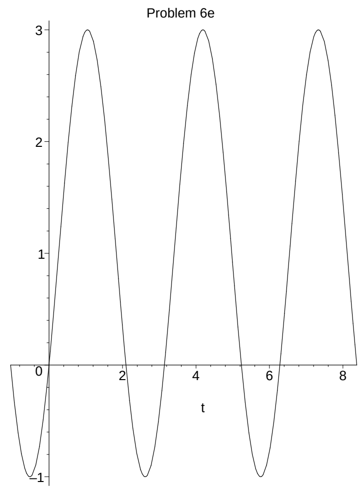
(c) What is the period of F?

Recall that the factor of 2 inside the cosine acts to compress the graph horizontally by a factor of 2. As the period of a regular cosine curve is 2π, the period of F is $\frac{2\pi}{2} = \pi$.

(d) What is the phase shift of F?

The factor of $\pi/3$ acts to translate the graph of F to the right by $\pi/3$, so it is the phase shift.

(e) Sketch a graphs of 3 periods of F.



7. Consider the function $F(t) = 3\tan(\pi(t - 1/2)) - 2$.

The reasoning of the last problem applies to this problem as well.

(a) What is the average value of F?

The average value is -2.

(b) What is the amplitude of F?

The amplitude is 3.

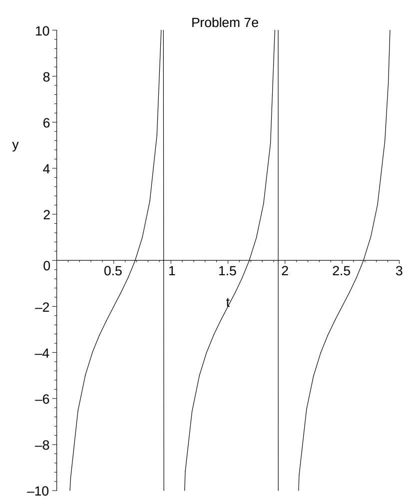
(c) What is the period of F?

The period is $\frac{\pi}{\pi} = 1$.

(d) What is the phase shift of F?

The phase shift is $1/2$.

(e) Sketch a graphs of 3 periods of F.

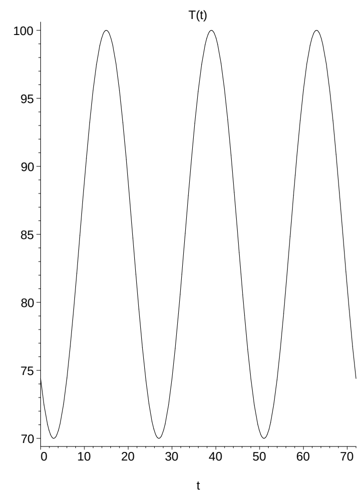


8. Suppose the temperature during an average July day has a highest value of 100, achieved at 3PM and that the lowest temperature is 70, achieved at 3AM. Let $T(t)$ be the function which measures the temperature T at time t, where t is the number of hours after midnight.

(a) What is the period of T?
The time difference between the highest and lowest temperature, which is half the period, is 12 hours. Therefore, the period is $2 \cdot 12 = 24$ hours.

(b) What is the average temperature and when does it occur?
The average temperature is the average of the high and low temperature, which is $1/2(100 + 70) = 85$. It occurs one quarter of a period (which is 6 hours) before and after the highest temperature occurs. Thus the average temperature occurs at 9AM and 9PM.

(c) Sketch the function $T(t)$ for two periods.



(d) Find a formula for T of the form $T(t) = a\cos(b(t - c)) + d$.
The period is $24 = 2\pi/b$, so $b = 2\pi/24 = \pi/12$. The average value is d, so $d = 85$. The amplitude of the temperature oscillations is the difference between the high temperature and the average temperature, which is $100 - 85 = 15$; so $a = 15$. The high temperature happens when $t = 15$, which happens when the thing inside the cosine is 0. Thus $\pi/12(15 - c) = 0$ and so $c = 15$. Putting this all together, we have

$$T(t) = 15\cos(\pi/12(t - 15)) + 85.$$

(e) Find a formula for T of the form $T(t) = A\sin(B(t - C)) + D$.
Solution #1: The period is $24 = 2\pi/B$, so $B = 2\pi/24 = \pi/12$. The average value is D, so $D = 85$. The amplitude of the temperature oscillations is the difference between the high temperature and the average temperature, which is $100 - 85 = 15$; so $A = 15$. The high temperature happens when $t = 15$, which happens when the thing inside the sine is $\pi/2$. Thus $\pi/12(15 - C) = \pi/2$ and so $C = 9$. Putting this all together, we have

$$T(t) = 15\sin(\pi/12(t - 9)) + 85.$$

Solution # 2:

$$\begin{aligned} T(t) &= 15\cos(\pi/12(t - 15)) + 85 = 15\sin(\pi/12(t - 15) + \pi/2) + 85 = 15\sin(\pi/12(t - 15) + 6\pi/12) + 85 \\ &= 15\sin(\pi/12(t - 15 + 6)) + 85 = 15\sin(\pi/12(t - 9)) + 85. \end{aligned}$$

(f) How many hours per day is the temperature below 65?
The low temperature is 70, which is strictly greater than 65. Thus the temperature is never below 65.

(g) How many hours per day is the temperature above 80?
Setting the temperature equal to 80, we have

$$80 = 15\cos(\pi/12(t - 15)) + 85 \qquad \text{or} \qquad -1/3 = \cos(\pi/12(t - 15)).$$

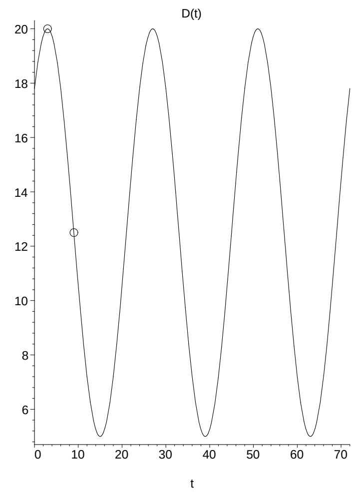
Taking the arccosine of both sides we obtain $\pm 1.911 = \pi/12(t - 15)$. So we have two times when the temperature is 80, namely

$$t_+ = 15 + \frac{1.911 \cdot 12}{\pi} \qquad t_- = 15 - \frac{1.911 \cdot 12}{\pi}.$$

The difference $t_+ - t_-$ is the time period we're looking for, and it is

$$t_+ - t_- = 15 + \frac{1.911 \cdot 12}{-}(15 - \frac{1.911 \cdot 12}{\pi}) = 14.60.$$

9. Suppose the depth of the water at the end of a pier is given by the graph below. The two marked points are $(3, 20)$ and $(9, 12.5)$.



(a) What is the period of D?
The two marked points are the highest value and the average value (respectively) of the depth. They occur 6 hours apart, which is one quarter of a period. So the period is $4 \cdot 6 = 24$ hours.

(b) What is the lowest depth and when does it occur?
The difference between the highest depth and the average depth is $20 - 12.5 = 7.5$, and that is the same as the difference between the average depth and the lowest depth. So the lowest depth is $12.5 - 7.5 = 5$. It occurs half a period (which is 12 hours) after the highest depth, or at $t = 15$ which is 3PM.

(c) Find a formula for D of the form $D(t) = a\cos(b(t - c)) + d$.
The period is $24 = 2\pi/b$, so $b = 2\pi/24 = \pi/12$. The average value is d, so $d = 12.5$. The amplitude of the depth oscillations is the difference between the high depth and the average depth, which is $20 - 12.5 = 7.5$; so $a = 7.5$. The high depth happens when $t = 3$, which happens when the thing inside the cosine is 0. Thus $\pi/12(3 - c) = 0$ and so $c = 3$. Putting this all together, we have

$$D(t) = 7.5\cos(\pi/12(t - 3)) + 12.5.$$

(d) Find a formula for D of the form $D(t) = A\sin(B(t - C)) + D$.
Solution #1: The period is $24 = 2\pi/B$, so $B = 2\pi/24 = \pi/12$. The average value is D, so $D = 12.5$. The amplitude of the depth oscillations is the difference between the high depth and the average depth, which is $20 - 12.5 = 7.5$; so $A = 7.5$. The high depth happens when $t = 3$, which happens when the thing inside the sine is $\pi/2$. Thus $\pi/12(3 - C) = \pi/2$ and so $C = -3$. Putting this all together, we have

$$D(t) = 7.5\sin(\pi/12(t + 3)) + 12.5.$$

Solution # 2:

$$\begin{aligned} D(t) &= 7.5\cos(\pi/12(t-3)) + 12.5 = 7.5\sin(\pi/12(t-3) + \pi/2) + 12.5 = 7.5\sin(\pi/12(t-3) + 6\pi/12) + 12.5 \\ &= 7.5\sin(\pi/12(t - 3 + 6)) + 12.5 = 7.5\sin(\pi/12(t+3)) + 12.5. \end{aligned}$$

(e) How many hours per day is the depth below 22?
The highest depth is 20, which is strictly less than 22. So the depth is always less than 22.

(f) How many hours per day is the depth above 15?
Setting the depth equal to 15, we have

$$15 = 7.5\cos(\pi/12(t-3)) + 12.5 \qquad \text{or} \qquad 1/3 = \cos(\pi/12(t-3)).$$

Taking the arccosine of both sides we obtain $\pm 1.231 = \pi/12(t - 3)$. So we have two times when the depth is 15, namely

$$t_+ = 3 + \frac{1.231 \cdot 12}{\pi} \qquad t_- = 3 - \frac{1.231 \cdot 12}{\pi}.$$

The difference $t_+ - t_-$ is the time period we're looking for, and it is

$$t_+ - t_- = 3 + \frac{1.231 \cdot 12}{\pi} - (3 - \frac{1.231 \cdot 12}{\pi}) = 9.404.$$

10. Find all the solutions to the equation $\sin(2\theta - \pi/3) = 1/2$ with

(a) $-\pi/2 \le \theta < \pi/2$
We know one solution from the arcsin button on a calculator: $2\theta - \pi/3 = \pi/6$. We can get another solution from a cofunction identity: $2\theta - \pi/3 = \pi - \pi/6 = 5\pi/6$. The general solutions are of the form

$$2\theta - \pi/3 = \pi/6 + 2k\pi \qquad 2\theta - \pi/3 = 5\pi/6 + 2k\pi,$$

which we can rearrange to read

$$\theta = \pi/4 + k\pi \qquad \theta = 7\pi/12 + k\pi.$$

Here k can be any integer. It remains to find which values of k will give us angles θ between $-\pi/2$ and $\pi/2$. In the first case, we have

$$-\pi/2 \le \pi/4 + k\pi < \pi/2 \qquad \text{or} \qquad -3/4 \le k < 1/4.$$

The only integer k which works is $k = 0$, and so $\theta = \pi/4$. In the other case we have

$$-\pi/2 \le 7\pi/12 + k\pi < \pi/2 \qquad \text{or} \qquad -13/12 \le k < 1/12.$$

The only integer k which works is $k = -1$, and so $\theta = -5\pi/12$.

(b) $3\pi/2 \le \theta < 5\pi/2$
Again we have the general form of θ:

$$\theta = \pi/4 + k\pi \qquad \theta = 7\pi/12 + k\pi.$$

It remains to find the values of k such that $3\pi/2 \le \theta < 5\pi/2$. In the first case we have

$$3\pi/2 \le \pi/4 + k\pi < 5\pi/2 \qquad \text{or} \qquad 5/4 \le k < 11/4.$$

The only integer which works is $k = 2$, and so $\theta = 9\pi/4$. In the other case we have

$$3\pi/2 \le 7\pi/12 + k\pi < 5\pi/2 \qquad \text{or} \qquad 11/12 \le 23/12.$$

The only integer which works is $k = 1$, and so $\theta = 19\pi/12$.

(c) $-2\pi \le \theta < -\pi$
Again we have the general form of θ:

$$\theta = \pi/4 + k\pi \qquad \theta = 7\pi/12 + k\pi.$$

It remains to find the values of k such that $-2\pi \le \theta < -\pi$. In the first case we have

$$-2\pi \le \pi/4 + k\pi < -\pi \qquad \text{or} \qquad -9/4 \le k < -5/4.$$

The only integer which works is $k = -2$, and so $\theta = -7\pi/4$. In the other case, we have

$$-2\pi \le 7\pi/12 + k\pi < -\pi \qquad \text{or} \qquad -33/12 \le k < -19/12.$$

The only integer which works is $k = -2$, and so $\theta = -17\pi/12$.

11. Find all the solutions to the equation $\tan(3\theta - \pi/2) = 1$ with

(a) $\pi/2 \le \theta < 3\pi/2$
We know one solution from the arctan button on a calculator: $3\theta - \pi/2 = \pi/4$. The general solution (using the periodicity of tangent) is given by

$$3\theta - \pi/2 = \pi/4 + k\pi \qquad \text{or} \qquad \theta = \pi/4 + k\pi/3.$$

Here k can be any integer. It remains to find the integers k such that $\pi/2 \le \theta < 3\pi/2$. We have

$$\pi/2 \le \pi/4 + k\pi/3 < 3\pi/2 \qquad \text{or} \qquad 3/4 \le k < 15/4.$$

The only integer which work are $k = 1, 2, 3$, and so $\theta = 7\pi/12, 11\pi/12, 5\pi/4$.

(b) $-3\pi/2 \le \theta < -\pi/2$
Again, the general form of θ is given by

$$3\theta - \pi/2 = \pi/4 + k\pi \qquad \text{or} \qquad \theta = \pi/4 + k\pi/3.$$

It remains to find the integers such that $-3\pi/2 \le \theta < -\pi/2$. We have

$$-3\pi/2 \le \pi/4 + k\pi/3 < -\pi/2 \qquad \text{or} \qquad -21/4 \le k < -9/4.$$

The only integers which work are $k = -5, -4, -3$, and so $\theta = -17\pi/12, -13\pi/12, -3\pi/4$.

(c) $\pi \le \theta < 2\pi$
Again, the general form of θ is given by

$$3\theta - \pi/2 = \pi/4 + k\pi \qquad \text{or} \qquad \theta = \pi/4 + k\pi/3.$$

It remains to find the integers such that $\pi \le \theta < 2\pi/$. We have

$$\pi \le \pi/4 + k\pi/3 < 2\pi \qquad \text{or} \qquad 9/4 \le k < 21/4.$$

The only integers which work are $k = 3, 4, 5$, and so $\theta = 5\pi/4, 19\pi/12, 23\pi/12$.

12. Find all the solutions to the equation $\cos^2\theta + 2\sin\theta = 1$ with

(a) $0 \le \theta < \pi$
First use the identity $\cos^2\theta + \sin^2\theta = 1$ to eliminate the $\cos^2\theta$ term in this equation:

$$1 = \cos^2\theta + 2\sin\theta = 1 - \sin^2\theta + 2\sin\theta.$$

We can rearrange this to read $0 = -\sin^2\theta + 2\sin\theta$, or

$$\sin^2\theta = 2\sin\theta.$$

Thus either $\sin\theta = 0$ or $\sin\theta = 2$; the latter case is impossible. In the former case, we know $\theta = 0$ is a solution. By a cofunction identity, $\theta = \pi - 0 = \pi$ is also a solution. The general solution is of the form

$$\theta = 0 + 2k\pi \qquad \text{or} \qquad \theta = \pi + 2k\pi.$$

Here k can be any integer. We can rearrange this to read $\theta = k\pi$. The only integer k such that θ lies in the right range is $k = 0$, and so $\theta = 0$ is the only solution.

(b) $-\pi/3 \le \theta < 2\pi/3$
Again, the general form of θ is $\theta = k\pi$, and only $k = 0$ works. So again $\theta = 0$ is the only solution.

(c) $\pi/2 \le \theta < 3\pi/2$
Again, the general form of θ is $\theta = k\pi$. This time, only $k = 1$ works, and so $\theta = \pi$ is the only solution.

13. Find all the solutions to the equation $\cos^2\theta - 2\sin\theta = 0$ with

(a) $-3\pi/2 \le \theta < -\pi/2$
Again, we will use the identity $\cos^2\theta + \sin^2\theta = 1$ to rearrange our equation as

$$0 = \cos^2\theta - 2\sin\theta = 1 - \sin^2\theta - 2\sin\theta,$$

or

$$\sin^2\theta + 2\sin\theta - 1 = 0.$$

Using the quadratic formula, we see

$$\sin\theta = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}.$$

The solution $\sin\theta = -1-\sqrt{2}$ is less than -1, which is impossible. So we only have the solution $\sin\theta = -1+\sqrt{2}$. Taking the arcsine of $-1+\sqrt{2}$ we find $\theta = .4271$. A cofunction identity implies $\theta = \pi - .4271 = 2.715$ is also a solution. Thus (using the periodicity of sin) we can write the general solution as

$$\theta = .4271 + 2k\pi \qquad \theta = 2.715 + 2k\pi.$$

Here k can be any integer. It remains to find the values of k such that $-3\pi/2 \le \theta < -\pi/2$. In the first case, we have

$$-3\pi/2 \le .4271 + 2k\pi < -\pi/2 \qquad \text{or} \qquad -3/4 - \frac{.4271}{2\pi} \le k < -1/4 - \frac{.4271}{2\pi}.$$

However, there are no integers k which satisfy both of these inequalities. In the other case, we have

$$-3\pi/2 \le 2.715 + 2k\pi < -\pi/2 \qquad \text{or} \qquad -3/4 - \frac{2.715}{2\pi} \le k < -1/4 - \frac{2.715}{2\pi}.$$

The only integer which works is $k = -1$, and so $\theta = 2.715 - 2\pi = -3.568$.

(b) $3\pi/2 \le \theta < 5\pi/2$
Again, the general for of θ is

$$\theta = .4271 + 2k\pi \qquad \theta = 2.715 + 2k\pi,$$

and it remains to find the values of k such that $3\pi/2 \le \theta < 5\pi/2$. In the first case we have

$$3\pi/2 \le .4271 + 2k\pi < 5\pi/2 \qquad \text{or} 3/4 - \frac{.4271}{2\pi} \le k < 5/4 - \frac{.4271}{2\pi}.$$

The only integer which works is $k = 1$, and so $\theta = .4271 + 2\pi = 6.710$. In the other case we have

$$3\pi/2 \le 2.715 + 2k\pi < 5\pi/2 \qquad \text{or} \qquad 3/4 - \frac{2.715}{2\pi} \le k < 5/4 - \frac{2.715}{2\pi}.$$

However there are no integers which satisfy both of these inequalities, so we don't have any more solutions.

(c) $\pi \le \theta < 2\pi$
Again, the general for of θ is

$$\theta = .4271 + 2k\pi \qquad \theta = 2.715 + 2k\pi,$$

and it remains to find the values of k such that $\pi \le \theta < 2\pi$. In the first case we have

$$\pi \le .4271 + 2k\pi < 2\pi \qquad \text{or} \qquad 1/2 - \frac{.4271}{2\pi} \le k < 1 - \frac{.4271}{2\pi}.$$

However, there are no integers which satisfy both of these inequalities. In the other case, we have

$$\pi \le 2.715 + 2k\pi < 2\pi \qquad \text{or} \qquad 1/2 - \frac{2.715}{2\pi} \le 1 - \frac{2.715}{2\pi}.$$

Again, there are no integers which satisfy these inequalities, so we find no solutions in this range.

14. Using the identities $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and $\cos^2\theta + \sin^2\theta = 1$ show

$$\cos(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}}.$$

First we have

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1.$$

Rearranging this, we have $\cos^2\theta = (1/2)(1 + \cos 2\theta)$. Taking the square root of both sides of this equation, we have

$$\cos\theta = \sqrt{\frac{1+\cos\theta}{2}}.$$

Replacing θ with $\theta/2$ in the above equation we obtain the identity we're looking for.

15. Suppose $\cos(u) = \sqrt{3}/2$ and $\sin(v) = -1/\sqrt{2}$. If u and v are both in the fourth quadrant, find

Before we start, it will be useful to find $\sin(u)$ and $\cos(v)$. First note $\sin(u) < 0$ and $\cos(v) > 0$ (as both angles are in the fourth quadrant). Using $\sin^2(u) = 1 - \cos^2(u) = 1 - 3/4$, we see $\sin(u) = -\sqrt{1/4} = -1/2$. Using $\cos^2(v) = 1 - sin^2(v) = 1 - 1/2$, we see $\cos(v) = \sqrt{1/2} = 1/\sqrt{2}$.

(a) $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v) = (\sqrt{3}/2)(1/\sqrt{2}) - (-1/2)(-1/\sqrt{2}) = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

(b) $\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v) = (-1/2)(1/\sqrt{2}) - (\sqrt{3}/2)(-1/\sqrt{2}) = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

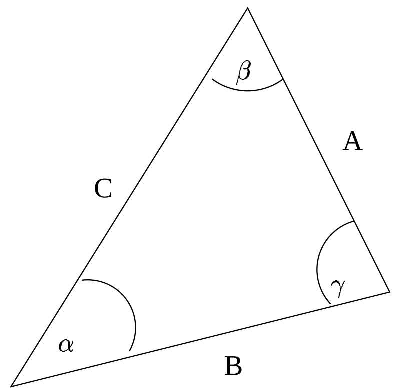
(c) $\tan(u+v) = \frac{\cos(u+v)}{\sin(u+v)} = \frac{cos(u)\cos(v) - \sin(u)\sin(v)}{\sin(u)\cos(v) + \cos(u)\sin(v)} = \frac{(\sqrt{3}/2)(1/\sqrt{2}) - (-1/2)(-1/\sqrt{2})}{(-1/2)(1/\sqrt{2}) + (\sqrt{3}/2)(-1/\sqrt{2})} = 1$

(d) $\cos(2u) = \cos^2(u) - \sin^2(u) = (\sqrt{3}/2)^2 - (-1/2)^2 = 1/2$.

(e) $\sin(2v) = 2\sin(v)\cos(v) = 2(-1/\sqrt{2})(1/\sqrt{2}) = -1$.

(f) $\sin(v/2) = \sqrt{(1/2)(1-\cos(v))} = \sqrt{(1/2)(1-1/\sqrt{2})} = \sqrt{1/2 - 1/(2\sqrt{2})}$

16. Consider the triangle drawn below.



In each case where you are given some of the information regarding the triangle (e.g. two angles and a side length) find the remaining side lengths and angles (when possible; if no solution exists, explain why and if multiple solutions exist find all of them). (Caution: the figure may not be drawn to scale. Also, it may help to reorient the triangle.)

(a) $\alpha = \pi/3$, $\beta = \pi/4$, $C = 10$

Here we will use the law of sines. First, $\gamma = \pi - \alpha - \beta = \pi - \pi/4 - \pi/3 = 5\pi/12$. Then

$$\frac{\sin(5\pi/12)}{10} = \frac{\sin(\pi/3)}{A},$$

and so $A = 8.966$. Finally,

$$\frac{\sin(5\pi/12)}{10} = \frac{\sin(\pi/4)}{B},$$

and so 7.321.

(b) $\alpha = \pi/6$, $\beta = \pi/2$, $A = 5$

Again, we will use the law of sines. First, $\gamma = \pi - \alpha - \beta = \pi - \pi/2 - \pi/6 = \pi/3$. Then

$$\frac{\sin(\pi/6)}{5} = \frac{\sin(\pi/2)}{B},$$

and so $B = 10$. Finally,

$$\frac{\sin(\pi/6)}{5} = \frac{\sin(\pi/3)}{C},$$

and so $C = 5\sqrt{3}$.

(c) $\alpha = \pi/3$, $A = 4$, $B = 3$

Again we use the law of sines. Incidentally, this is the SSA case where we have a unique triangle, because the side opposite the angle we're given (namely A) is larger than the other side we're given (namely B). Anyhow, we have

$$\frac{\sin(\pi/3)}{4} = \frac{\sin\beta}{3},$$

and so $\sin\beta = 3 = sqrt3/8$ and $\beta = .7070$. Then $\gamma = \pi - \alpha - \beta = \pi - \pi/3 - .7070 = 1.387$. Finally,

$$\frac{\sin(\pi/3)}{4} = \frac{\sin(.7070)}{C},$$

and so $C = 3.000$.

(d) $A = 3$, $B = 5$, $C = 7$

This time we will use the law of cosines. First,

$$3^2 = 5^2 + 7^2 - 2 \cdot 5 \cdot 7 \cos(\alpha),$$

and so $\cos(\alpha) = 13/14$ and $\alpha = .3802$. Next,

$$5^2 = 3^2 + 7^2 - 2 \cdot 3 \cdot 7 \cos(\beta),$$

and so $\cos(\beta) = 11/14$ and $\beta = .6670$. Finally, $\gamma = \pi - \alpha - \beta = \pi - .3802 - .6670 = 2.094$.

(e) $A = 3$, $B = 5$, $\gamma = \pi/3$

Again we will use the law of cosines:

$$C^2 = A^2 + B^2 - 2AB \cos\gamma = 3^2 + 5^2 - 30\cos(\pi/3) = 19,$$

and so $C = \sqrt{19}$. Then by the law of sines,

$$\frac{\sin\alpha}{3} = \frac{\sin(\pi/3)}{\sqrt{19}},$$

and so $\sin\alpha = \frac{3\sqrt{3}}{2\sqrt{19}}$ and $\alpha = .6386$. Finally, $\beta = \pi - \alpha - \gamma = \pi - \pi/3 - .6386 = 1.456$.

(f) $\alpha = \pi/4$, $A = 3$, $B = 4$.

This is the case where we don't have any triangles with this data: $B\sin\alpha = 4\sin(\pi/4) = 4/\sqrt{2} < 3 = A$.

17. Find the area of the triangles with the following properties. Sides and angles are labeled as in the previous problem.

(a) base of 10 and height of 5

The are is one half the base times the height, which in this case is $\frac{1}{2} \cdot 5 \cdot 10 = 25$.

(b) $A = 10$, $B = 12$ and $\gamma = \pi/4$

Here we use the fact that the area is $\frac{1}{2}AB\sin\gamma = \frac{1}{2}(10)(12)\sin(\pi/4) = 60/\sqrt{2}$

(c) $A = 10$, $B = 8$ and $\beta = \pi/3$

It turns out in this case that there are no triangles with this data. To see this, use the law of sines:

$$\frac{\sin\alpha}{10} = \frac{\sin(\pi/3)}{8},$$

or $\sin\alpha = \frac{5\sqrt{3}}{8} > 1$.

(d) $A = 10$, $B = 8$ and $C = 6$

Here we use Heron's formula:Area $= \sqrt{s(s-A)(s-B)(s-C)}$. In this case, $s = \frac{10+8+6}{2} = 12$, so the area is $\sqrt{12(12-10)(12-8)(12-6)} = \sqrt{576} = 24$.

(e) $A = 10$, $\alpha = \pi/3$ and $\beta = \pi/4$

We want to find the side length C and use the fact that the area is $\frac{1}{2}AC\sin\alpha$. First, we know that the angle γ is $\pi - \pi/3 - \pi/4 = 5\pi/12$. Then using the law of sines we have

$$\frac{\sin(\pi/3)}{10} = \frac{\sin(5\pi/12)}{C},$$

or $C = 9.028$. Then the area is $\frac{1}{2}AC\sin\beta = \frac{1}{2}(10)(9.028)(1/\sqrt{2}) = 31.92$.

(f) $A = 10$, $\gamma = \pi/3$ and $\beta = \pi/4$.

We want to find the side length B and use the fact that the area is $\frac{1}{2}AB\sin\gamma$. First, we know that $\alpha = \pi - \pi/3 - \pi/4 = 5\pi/12$. Then using the law of sines we have

$$\frac{\sin(5\pi/12)}{10} = \frac{\sin(\pi/4)}{B},$$

or $B = 7.321$. Then the area is $\frac{1}{2}(10)(7.321)\sin(\pi/3) = 31.70$.

(g) $A = 10$, $B = 8$ and $C = 12$

Here we use Heron's formula:Area $= \sqrt{s(s-A)(s-B)(s-C)}$. In this case, $s = \frac{10+8+12}{2} = 15$, so the area is $\sqrt{15(15-10)(15-8)(15-12)} = \sqrt{1575} = 5\sqrt{63}$.

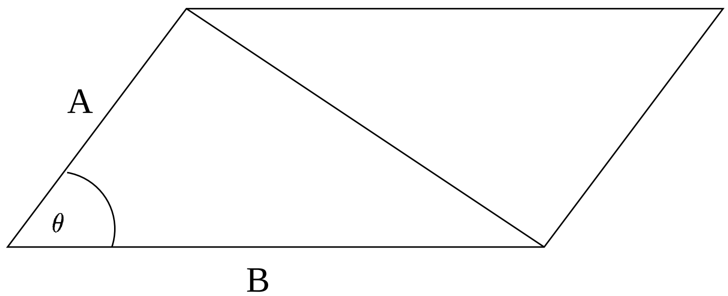
(h) $A = 10$, $\beta = \pi/4$ and $C = 8$

Here we use that the area is $\frac{1}{2}AC\sin\beta = \frac{1}{2}(10)(8)\sin(\pi/4) = 40/\sqrt{2}$.

18. Show that the area of the parallelogram shown below is $AB\sin\theta$.

Bisect the parallelogram with a diagonal line as shown. You obtain two triangles which are congruent (by SAS). Each triangle has area $\frac{1}{2}AB\sin\theta$. Therefore, the area of the parallelogram is the sum of the areas of the triangles, which is

$$\frac{1}{2}AB\sin\theta + \frac{1}{2}AB\sin\theta = AB\sin\theta.$$



19. Suppose you know two side lengths and an angle of a triangle. Can you determine the area of the triangle? Justify your answer either way (i.e. either explain why you can find the area or find a counterexample where you can't).

No. Consider a triangle where you're given $A = 5$, $B = 4$ and $\beta = \pi/6$. Then you want to find one other side (say C), so that you can use that the area is $\frac{1}{2}AC\sin\beta$. First you need to use the law of sines to find α:

$$\frac{\sin(\pi/6)}{4} = \frac{\sin\alpha}{5},$$

or $\sin\alpha = 5/8$. Then we have two solutions for α: either $\alpha = .6751$ or $\alpha = \pi - .6751 = 2.466$. In the former case, $\gamma = \pi - \pi/6 - .6751 = 1.943$. In the latter case, $\gamma = \pi - \pi/6 - 2.466 = .1520$. We can use the law of sines to find C:

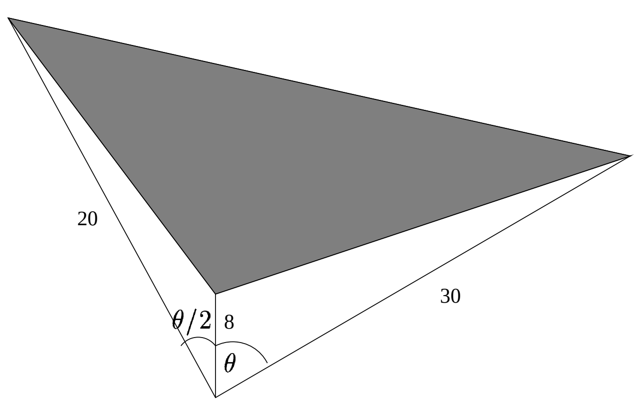
$$\frac{\sin(\pi/6)}{4} = \frac{\sin\gamma}{C}.$$

The two case we have yield either $C = 7.452$ and the area is 18.63 or $C = 1.211$ and the area is 3.028.

20. Suppose you are given three pieces of information regarding a triangle (e.g. two angles and a side length). What is the maximum number of triangles which satisfy the conditions you're given (i.e. have the same information as you're given)? Explain your answer.

If you are given the three angles of the triangle, you could have infinitely many such triangles. Pick one side length and call it A. By rescaling the triangle you can make that side length any positive number you please, without changing any of the angles.

21. Consider the three triangles drawn below.



(a) Find the area of the shaded triangle as a function of θ.

The area of the shaded region is the area of the big triangle minus the area of the two small unshaded triangles. First, the area of the big triangle is

$$\frac{1}{2} \cdot 20 \cdot 30 \sin(3\theta/2) = 300 \sin(3\theta/2).$$

Next, the area of the small triangle on the left is

$$\frac{1}{2} \cdot 20 \cdot 8 \sin(\theta/2) = 80 \sin(\theta/2).$$

The area of the small triangle on the right is

$$\frac{1}{2} \cdot 8 \cdot 30 \sin(\theta) = 120 \sin(\theta).$$

Putting this all together, we see that the shaded area is

$$300 \sin(3\theta/2) - 80 \sin(\theta/2) - 120 \sin(\theta).$$

(b) For which values of θ is this area positive?

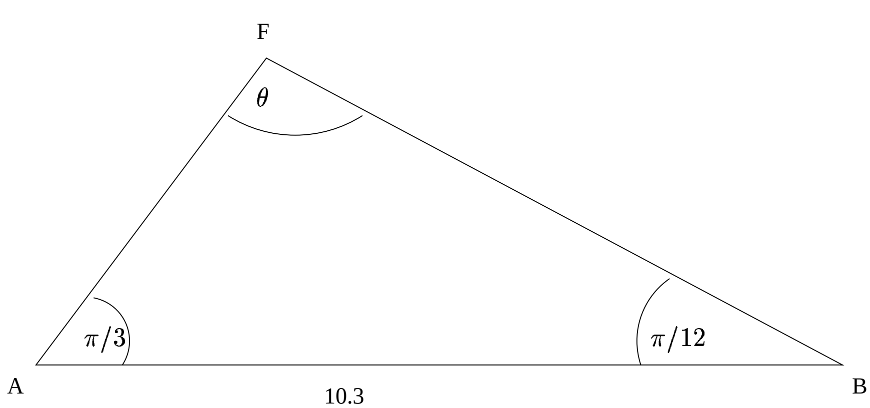
The easiest way to think about this problem is to imagine varying the angle θ starting at $\theta = 0$. This amounts to opening a hinge with sides 20 and 30. Anyhow, when $\theta = 0$ everything collapses to a line segment of length 30, but as soon as θ becomes positive the shaded area is positive. How much can you increase θ and still have the shaded area? Well, the shaded area will disappear when the line segment of length 8 meets the long side of the big triangle, which it does at a right angle. Then you get a right triangle with hypotenuse 30 and where the side adjacent to θ has length 8. Then $\cos\theta = 8/30 = 4/15$ and $\theta = 1.301$.

22. Use the law of cosines ($A^2 = B^2 + C^2 - 2BC\cos\alpha$, where the triangle is again labeled as in problem 16) to show

$$\frac{1}{2}BC(1 - \cos\alpha) = \frac{A+B-C}{2} \cdot \frac{A-B+C}{2}.$$

$$\begin{aligned} \frac{A+B-C}{2} \cdot \frac{A-B+C}{2} &= \frac{(A+B-C)(A-B+C)}{4} = \frac{A^2 - AB + AC + AB - B^2 + BC - AC + BC - C^2}{4} \\ &= \frac{2BC + A^2 - B^2 - C^2}{4} = \frac{2BC - 2BC\cos\alpha}{4} \\ &= \frac{BC - BC\cos\alpha}{2} = \frac{BC(1-\cos\alpha)}{2} = \frac{1}{2}BC(1-\cos\alpha) \end{aligned}$$

23. A fire at location F is spotted from two fire stations, A and B, which are 10.3 miles apart. If the angle ABF is $\pi/12$ and the angle BAF is $\pi/3$ find the distance from the fire to each of the fire stations.



First we find the remaining angle θ: $\theta = \pi - \pi/3 - \pi/12 = 7\pi/12$. Then distance from A to F is given by the law of sines:

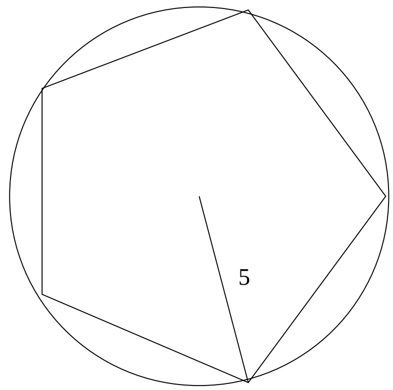
$$\frac{\sin(\pi/12)}{AF} = \frac{\sin(7\pi/12)}{10.3},$$

so $AF = 2.760$ miles. Similarly, we can find the distance from B to F using the law of sines:

$$\frac{\sin(\pi/3)}{BF} = \frac{\sin(7\pi/12)}{10.3},$$

so $BF = 9.235$ miles.

24. A regular pentagon is a five sided polygon such that all the interior angles are equal. Find the perimeter (i.e. the length of the boundary curve) of a regular pentagon inscribed in a circle of radius 5 (see the figure below).



Draw five radial line segments, connecting the center of the circle to each of the five vertices of the pentagon; you will obtain five congruent triangles inside the pentagon. Call the center angle of each of the triangles θ (i.e. θ is the angle of the triangle at the center of the circle). Because the pentagon is regular, $5\theta = 2\pi$, or $\theta = 2\pi/5$. Moreover, the remaining two angles are opposite radial line segments, so they have to be equal. Call their common value ϕ. So $\pi = 2\pi/5 + 2\phi$, or $\phi = 3\pi/10$. Now let A denote the length of one of the sides of the pentagon (this is also the side opposite the angle $2\pi/5$ on the triangle). By the law of sines,

$$\frac{\sin(2\pi/5)}{A} = \frac{\sin(3\pi/10)}{5},$$

which we can rearrange to read

$$A = \frac{5\sin(2\pi/5)}{\sin(3\pi/10)} = 5.878.$$

The perimeter of the pentagon is $5A = 29.39$.