Solutions to the Practice Problems

Math 1060

November 1, 2002

1. Consider the function $F(t) = 1 - \cos(\pi/2(t-3))$.

   (a) What is the period of F?

   Recall that the factor of $\pi/2$ acts as a horizontal compression factor on the graph: i.e. this factor acts to compress the graphs in the horizontal direction by a factor of $\pi/2$. The of a standard cosine curve is 2π, so the period of F is $\frac{2\pi}{\pi/2} = 4$.

   (b) What is the amplitude of F?

   Recall that the -1 in front of the cosine acts to reflect the graph through the horizontal axis, but it doesn't change the size of the oscillations of the cosine curve. So the amplitude remains 1.

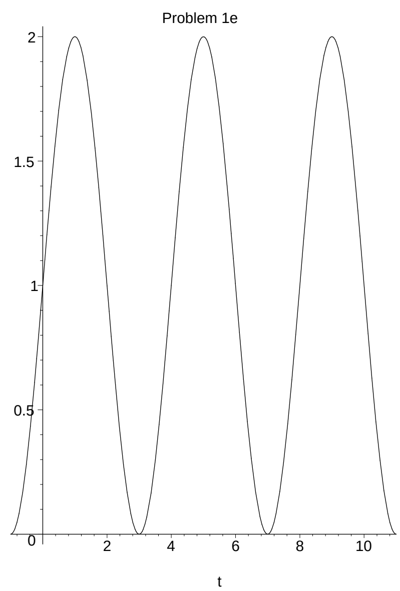
   (c) What is the average value of F?

   Recall that adding 1 to cosine acts to translate the graph up by 1. A standard cosine curve oscillates about an average value of 0, so this translated curve oscillates about an average value of 1.

   (d) What is the phase shift of F?

   The factor of 3 acts as a horizontal translation, so this is the phase shift. Another way to see this is to recognize that the trough (i.e. minimum value) of $1 - \cos((\pi/2)t)$ occurs at $t = 0$. So the minimum value of $1 - \cos(\pi/2(t-3))$ occurs when $\pi/2(t-3) = 0$, or when $t = 3$.

   (e) Sketch the graph of F for three periods.



2. Consider the function $F(t) = \frac{3}{2}\sin(3(t - \pi/5)) - 3$.

   The reasoning of the previous problem applies to this problem as well.

   (a) What is the period of F?

   The period is $\frac{2\pi}{3}$.

   (b) What is the amplitude of F?

   The amplitude is $3/2$.

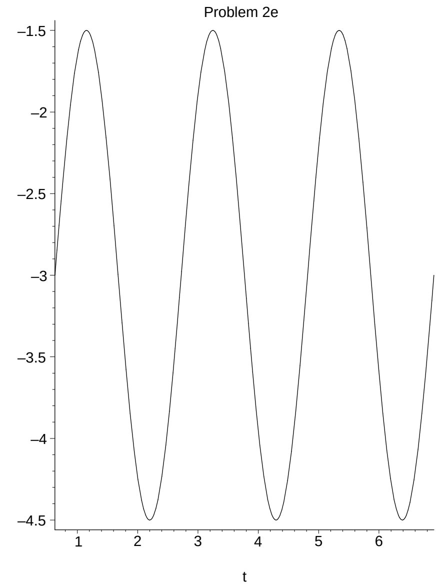
   (c) What is the average value of F?

   The average value is 3.

   (d) What is the phase shift of F?

   The phase shift is $\pi/5$.

   (e) Sketch the graph of F for three periods.



3. Consider the function $F(t) = 2\tan(2(t - \pi/3)) + 1$.

   The reasoning of the first problem applies to this problem as well.

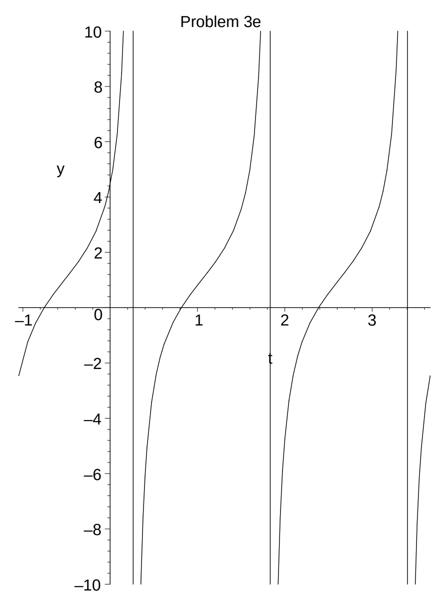
   (a) What is the period of F?
   The period is $\pi/2$ (recall that the period of a standard tangent curve is π).

   (b) What is the amplitude of F?
   The amplitude is 2.

   (c) What is the average value of F?
   The average value is 1.

   (d) What is the phase shift of F?
   The phase shift is $\pi/3$.

   (e) Sketch the graph of F for three periods.



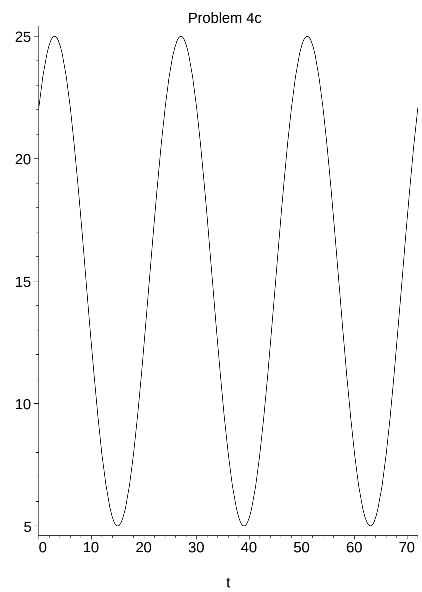
4. Suppose the depth of the water at the end of a dock varies periodically, with one period per 24 hours. Suppose also that the peak depth of 25 feet occurs at 3AM and that the average depth is 15 feet. Let $D(t)$ represent the depth of the water at time t, where t measures hours after midnight.

   (a) What is the lowest depth of the water and when does it occur?
   The peak depth is 25 and the average depth is 15, which means that the size of the oscillations of the depth is 10. Thus the minimum depth is $15 - 10 = 5$ feet. Also, this minimum depth occurs half a period after the maximum depth, or a $t = 3 + 12 = 15$, which is 3PM.

(b) When does the depth achieve its average value?

The average depth occurs at a quarter of a period after the both the maximum and minimum depths, which is $t = 3 + 6 = 9$ (i.e. 9 AM) and $t = 15 + 6 = 21$ (i.e. 9PM).

(c) Sketch the graph of D for three periods.



(d) Find a formula for $D(t)$ of the form $D(t) = a\cos(b(t - c)) + d$.

We know that the average value is 15, so $d = 15$. The size of the oscillations is 10, so $a = 10$. The period is 24, so $b = \frac{2\pi}{24} = \pi/12$. Finally, the maximum depth occurs at 3AM, which means $0 = \pi/12(3 - c)$, or $c = 3$. Putting this all together, we see

$$D(t) = 10\cos(\pi/12(t - 3)) + 15.$$

(e) Find a formula for $D(t)$ of the form $D(t) = A\sin(B(t - C) + D$. (There are at least two ways to do this.)

Solution # 1: First, the average value is 15, so again $D = 15$. The size of the oscillations is 10, so $A = 10$. The period is 24, so $B = \frac{2\pi}{24} = \pi/12$. Finally, the average value before the first peak occurs at $t = -3$ (i.e. 9PM the previous night), so $C = -3$. Putting this all together, we see

$$D(t) = 10\sin(\pi/12(t + 3) + 15.$$

Solution # 2: Using $\cos(\theta) = \sin(\theta + \pi/2)$, we see

$$\begin{aligned} D(t) &= 10\cos(\pi/12(t - 3)) + 15 = 10\sin(\pi/12(t - 3) + \pi/2) + 15 \\ &= 10\sin(\frac{\pi}{12}(t - 3) + \frac{6\pi}{12}) + 15 = 10\sin(\pi/12(t - 3 + 6)) + 15 \\ &= 10\sin(\pi/12(t + 3)) + 15 \end{aligned}$$

(f) Suppose a boat need a depth of at least 10 feet in order to moor to the dock. When can the boat moor?

We want to find the hours during the day when the depth of the water is at least 10 feet. Setting $D(t) = 10$, we find

$$10 = 10\cos(\pi/12(t - 3)) + 15 \qquad \text{or} \qquad -1/2 = \cos(\pi/12(t - 3)).$$

We have two solutions to this equation, namely

$$\frac{2\pi}{3} = \frac{\pi}{12}(t_+ - 3) \qquad \text{and} \qquad -\frac{2\pi}{3} = \frac{\pi}{12}(t_- - 3).$$

These two times correspond to the time when the depth first rises above 10 feet (this is t_-) and the time when the depth falls below 10 feet (this is t_+). (It is always easiest to tell which times are which by looking at the graph of D.) Solving for $t_\pm$, we find $t_+ = 11$ and $t_- = -5$. The difference $t_+ - t_- = 11 - (-5) = 16$ is the time interval we're looking for.

(g) Suppose there are some rocks which are exposed when the depth of the water is less than 7 feet. When are the rocks exposed?

Again, we set $D(t) = 7$ to get

$$7 = 10\cos(\pi/12(t-3)) + 15 \qquad \text{or} \qquad -\frac{4}{5} = \cos(\pi/12(t-3)).$$

We have two solutions to this equation, namely

$$2.4981 = \frac{\pi}{12}(t_1 - 3) \qquad \text{and} \qquad 3.7851 = \frac{\pi}{12}(t_2 - 3).$$

These times correspond to the times when the depth of the water first falls below 7 feet (this is t_1) and when the depth of the water rises back up to 7 feet (this is t_2). Solving for t_1 and t_2 we find

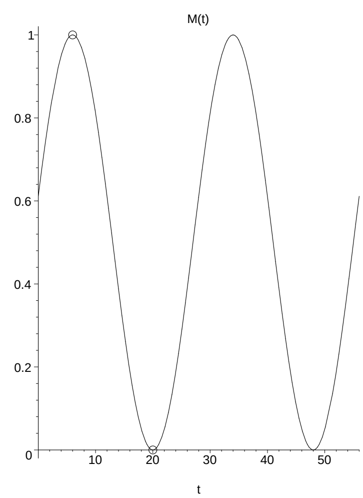
$$t_1 = 12.542 \qquad \text{and} \qquad t_2 = 17.458.$$

The difference $t_2 - t_1 = 4.916$ is the time interval we're looking for.

(h) Suppose there is some buried treasure which is exposed when the depth of the water is less than 3 feet. When is the buried treasure exposed?

Notice that the minimum depth of the water is 5 feet, so the buried treasure is never exposed.

5. Suppose the phase of the moon is given by the following graph. Let M(t) be the percentage of the moon illuminated on day t, where t represents days after Jan. 1.



The two marked points are $(6, 1)$ and $(20, 0)$.

(a) What is the period of M?

The two marked points are consecutive maximum and minimum points, respectively, and so they are half a period apart. Thus the period is $2(20 - 6) = 28$.

(b) What is the amplitude of M?

The amplitude is 1/2 the difference of the maximum and minimum values, which is $1/2(1 - 0) = 1/2$.

(c) What is the average value of M?

The average value is the average of the maximum and minimum values, which is $1/2(1 + 0) = 1/2$.

(d) What is the phase shift of M?

The peak of a normal cosine curve occurs at $t = 0$, whereas the peak of this curve occurs at $t = 6$. Therefore this curve is translated 6 to the right, so the phase shift is 6.

(e) Find a formula for M of the form $M(t) = a\cos(b(t - c)) + d$.

We know that the average value is 1/2, so $d = 1/2$. The size of the oscillations is also 1/2, so $a = 1/2$. The period is 28, so $b = \frac{2\pi}{28} = \pi/14$. Finally, the maximum depth occurs at day 6m, which means $0 = \pi/14(6 - c)$, or $c = 6$. Putting this all together, we see

$$M(t) = 1/2\cos(\pi/14(t-6)) + 1/2.$$

(f) Find a formula for M of the form $M(t) = A\sin(B(t-C)) + D)$.
Solution # 1: First, the average value is 1/2, so again $D = 1/2$. The size of the oscillations is 1/2, so $A = 1/2$. The period is 28, so $B = \frac{2\pi}{28} = \pi/114$. Finally, the average value before the first peak occurs at day $6 - 7 = -1$, so $C = -1$. Putting this all together, we see

$$M(t) = 1/2\sin(\pi/14(t+1)) + 1/2.$$

Solution # 2: Using $\cos(\theta) = \sin(\theta + \pi/2)$, we see

$$\begin{aligned} M(t) &= 1/2\cos(\pi/14(t-6)) + 1/2 = 1/2\sin(\pi/14(t-6) + \pi/2) + 1/2 \\ &= 1/2\sin(\frac{\pi}{14}(t-6) + \frac{7\pi}{14}) + 1/2 = 1/2\sin(\pi/14(t-3+7)) + 1/2 \\ &= 1/2\sin(\pi/14(t+1)) + 1/2 \end{aligned}$$

(g) How many days per cycle is the percentage of the moon illuminated at least 3/4?
We want to find the number of days where the moon's illumination is at least 3/4. Setting $M(t) = 3/4$ we get

$$3/4 = 1/2\cos(\pi/14(t-6)) + 1/2 \qquad \text{or} \qquad 1/2 = \cos(\pi/14(t-6)).$$

We have two solutions to this equation, namely

$$\frac{\pi}{3} = \frac{\pi}{14}(t_+ - 6) \qquad \text{and} \qquad -\frac{\pi}{3} = \frac{\pi}{14}(t_- - 6).$$

These two times correspond to the first time the moon's illumination goes above 3/4 (this is t_-) and the last it it above 3/4 (this is t_+). Solving for $t_\pm$, we find $t_+ = \frac{14}{3} + 6$ and $t_- = \frac{14}{3} + 6$. The difference $t_+ - t_- = \frac{28}{3}$ is the time interval we're looking for.

(h) How many days per cycle is the percentage of the moon illuminated at least 1/4.
Solution # 1: Again, we set $M(t) = 1/4$ to get

$$1/4 = 1/2\cos(\pi/14(t-6)) + 1/2 \qquad \text{or} \qquad -1/2 = \cos(\pi/14(t-6)).$$

We have two solutions to this equation, namely

$$\frac{2\pi}{3} = \frac{\pi}{14}(t_+ - 6) \qquad \text{and} \qquad -\frac{2\pi}{3} = \frac{\pi}{14}(t_- - 6).$$

These two times correspond to the time when the illumination first creeps above 1/4 (this is t_-) and the time when the illumination falls below 1/4 (this is t_+). Solving for $t_\pm$ we find $t_+ = \frac{28}{3} + 6$ and $t_- = -\frac{28}{3} + 6$. The difference $t_+ - t_- = \frac{56}{3}$ is the time interval we're looking for.
Solution # 2: Let T_1 be the time the moon's illumination is at least 1/4 and let T_2 be the time the moon's illumination is less than 1/4. We know that $T_1 + T_2 = 28$ (i.e. one full period). Now turn the graph of M upside-down; this will swap an illumination level of 1/4 with an illumination level of 3/4. Therefore, T_2, which is the time the moon's illumination is less than 1/4 is also the time the moon's illumination is at least 3/4. We've already found this last quantity, so we know that $T_2 = \frac{28}{3}$. Therefore

$$T_1 = 28 - T_2 = 28 - \frac{28}{3} = \frac{56}{3}.$$

6. Find all solutions θ to the equation $\tan(2\theta - \pi/3) = 1/3$ where

(a) $-\pi/2 \le \theta < \pi/2$
We can obtain one solution using the arctangent button on a calculator: $2\theta - \pi/3 = \arctan(1/3) = .3218$. Using the fact that tan is π-periodic, we see

$$2\theta - \pi/3 = .3218 + \pi k \qquad \text{or} \qquad \theta = \frac{1.3689 + \pi k}{2}.$$

Here k is any integer. It remains to find the values of k such that the angle θ lies between $-\pi/2$ and $\pi/2$. We want

$$-\frac{\pi}{2} \le \frac{1.3689 + \pi k}{2} < \frac{\pi}{2},$$

which we can rearrange to read

$$-1 - \frac{1.3689}{\pi} \le k < 1 - \frac{1.3689}{\pi}.$$

Thus we can have $k = -1$ and $k = 0$ and our solutions are

$$\theta = \frac{1.3689}{2}, \frac{1.3689 - \pi}{2}.$$

(b) $\pi/2 \le \theta < 3\pi/2$

Again, we have $\theta = \frac{1.3689+\pi k}{2}$ and we have to find the values of k such that θ lies in the correct range. This time we have

$$\frac{\pi}{2} \le \frac{1.3689+\pi k}{2} < \frac{3\pi}{2},$$

which we can rearrange to read

$$1 - \frac{1.3689}{\pi} \le k < 3 - \frac{1.3689}{\pi}.$$

Thus we have $k = 1, 2$ and our solutions are

$$\theta = \frac{1.3689+\pi}{2}, \frac{1.3689+2\pi}{2}.$$

(c) $-2\pi \le \theta < -\pi$

Again, we have $\theta = \frac{1.3689+\pi k}{2}$ and we have to find the values of k such that θ lies in the correct range. This time we have

$$2\pi \le \frac{1.3689+\pi k}{2} < 3\pi,$$

which we can rearrange to read

$$4 - \frac{1.3689}{\pi} \le k < 6 - \frac{1.3689}{\pi}.$$

Thus we have $k = 4, 5$ and our solutions are

$$\theta = \frac{1.3689+4\pi}{2}, \frac{1.3689+5\pi}{2}.$$

7. Find all solutions θ to the equation $\sin(3\theta - \pi/4) = 1/5$ where

(a) $-\pi/2 \le \theta < \pi/2$

We can obtain one solution using the arcsin button on a calculator: $3\theta - \pi/4 = \arcsin(1/5) = .2014$. Using the fact that sin is 2π-periodic, we also get the solutions

$$3\theta - \pi/4 = .2014 + 2\pi k \qquad \text{or} \qquad \theta = \frac{.9868+2\pi k}{3}.$$

Here k is any integer. However, these are not all the solutions. We also need to use the fact that $\sin(x) = \sin(\pi - x)$, and so we also have $3\theta - \pi/4 = \pi - .2014 = 2.940$. Using periodicity again, we have

$$3\theta - \pi/4 = 2.940 + 2\pi k \qquad \text{or} \qquad \theta = \frac{3.7256+2\pi k}{3}.$$

It remains to find the values of k which give us an angle between $-\pi/2$ and $\pi/2$. In the first case we have

$$-\frac{\pi}{2} \le \frac{.9868+2\pi k}{3} < \frac{\pi}{2},$$

which we can rearrange to read

$$-\frac{3}{4} - \frac{.9868}{2\pi} \le k < \frac{3}{4} - \frac{.9868}{2\pi}.$$

The only integer which satisfies these inequalities is $k = 0$, and so we obtain the solution $\theta = \frac{.9868}{3}$. In the second case we have

$$-\frac{\pi}{2} \le \frac{3.7256+2\pi k}{3} < \frac{\pi}{2},$$

which we can rearrange to read

$$-\frac{3}{4} - \frac{3.7256}{2\pi} \le k < \frac{3}{4} - \frac{3.7256}{2\pi}.$$

The integers $k = -1$ and $k = 0$ work for these inequalities, so we have the solutions $\theta = \frac{3.7256}{2\pi}, \frac{3.7256-2\pi}{2\pi}$.

(b) $-3\pi/2 \le \theta < -\pi/2$

Again, we have the two cases:

$$\theta = \frac{.9868+2\pi k}{3} \qquad \theta = \frac{3.7256+2\pi k}{3}.$$

To satisfy the bounds in the first case, we have

$$-\frac{3\pi}{2} \leq \frac{.9868 + 2\pi k}{3} < -\frac{\pi}{2},$$

which we can rearrange to read

$$-\frac{9}{4} - \frac{.9868}{2\pi} \leq k < -\frac{3}{4} - \frac{.9868}{2\pi}.$$

Therefore, $k = -2, -1$ and $\theta = \frac{.9868-2\pi}{3}, \frac{.9868-4\pi}{3}$. In the second case we have

$$-\frac{3\pi}{2} \leq \frac{3.7256 + 2\pi k}{3} < -\frac{\pi}{2},$$

which we can rearrange to read

$$-\frac{9}{4} - \frac{3.7256}{2\pi} \leq k < -\frac{3}{4} - \frac{3.7256}{2\pi}.$$

Thus $k = -2$ and $\theta = \frac{3.7256-4\pi}{3}$.

(c) $\pi \leq \theta < 2\pi$

Again, we have the two cases:

$$\theta = \frac{.9868 + 2\pi k}{3} \qquad \theta = \frac{3.7256 + 2\pi k}{3}.$$

To satisfy the bounds in the first case, we have

$$\pi \leq \frac{.9868 + 2\pi k}{3} < 2\pi,$$

which we can rearrange to read

$$\frac{3}{2} - \frac{.9868}{2\pi} \leq k < 3 - \frac{.9868}{2\pi}.$$

Then $k = 2$ and so $\theta = \frac{.9868+4\pi}{3}$. In the second case we have

$$\pi \leq \frac{3.7256 + 2\pi k}{3} < 2\pi,$$

which we can rearrange to read

$$\frac{3}{2} - \frac{3.7256}{2\pi} \leq k < 3 - \frac{3.7256}{2\pi}.$$

Thus $k = 1, 2$ and so $\theta = \frac{3.7256+2\pi}{3}, \frac{3.7256+4\pi}{3}$.

8. Find all solutions θ to the equation $\sin^2\theta + \frac{1}{3}\cos\theta = 1$ where

(a) $0 \leq \theta < 2\pi$

The first thing we want to do is replace $\sin^2\theta$ with $1 - \cos^2\theta$. Now the equation reads

$$1 - \cos^2\theta + \frac{1}{3}\cos\theta = 1 \qquad \text{or} \qquad -\cos^2\theta + \frac{1}{3}\cos\theta = 0.$$

Factoring out $-\cos\theta$, we get

$$-\cos\theta(\cos\theta - \frac{1}{3}) = 0.$$

Thus we either have $\cos\theta = 0$ or $\cos\theta = 1/3$.

In the first case, we have $\cos\theta = 0$, and so $\theta = \pi/2, 3\pi/2$. In the second case, we have $\cos\theta = 1/3$, and so $\theta = 1.23095, 2\pi - 1.23095 = 5.05226$.

(b) $-2\pi \leq \theta < -\pi$

We can use the answers from the previous part and the fact that $\cos\theta$ is 2π periodic. For the case where $\cos\theta = 0$, we have $\theta = \pi/2 - 2\pi = -3\pi/2$. For the case where $\cos\theta = 1/3$ we have $\theta = 1.23095 - 2\pi = -5.05226$.

(c) $\pi/2 \leq \theta < 3\pi/2$

Again, we will use our previous answers and the periodicity of $\cos\theta$. For the case where $\cos\theta = 0$ we have $\theta = \pi/2$, and the case where $\cos\theta = 1/3$ doesn't give us any solutions between $\pi/2$ and $3\pi/2$.

9. Find all the solutions θ to the equation $\cos^2\theta - \frac{1}{4}\sin\theta = 0$ where

(a) $0 \le \theta < 2\pi$
Using $\cos^2\theta = 1 - \sin^2\theta$, we can rearrange this equation as follows:

$$0 = \cos^2\theta - \frac{1}{4}\sin\theta = 1 - \sin^2\theta - \frac{1}{4}\sin\theta \qquad \text{or} \qquad 0 = \sin^2\theta + \frac{1}{4}\sin\theta - 1.$$

Using the quadratic formula, we obtain

$$\sin\theta = \frac{-1/4 \pm \sqrt{1/16+4}}{2} = \frac{-1/4 \pm \sqrt{1/16+64/16}}{2} = \frac{-1/4 \pm \sqrt{65/16}}{2} = -\frac{1}{8} \pm \frac{\sqrt{65}}{8}.$$

There are no solutions to the equation $\sin\theta = -1/8 - \sqrt{65}8$, because this number is less than -1. However, we can find solutions to the equation $\sin\theta = -1/8 + \sqrt{65}/8 = .88278$. Using the arsine button on a calculator we get the solution $\theta = .77251$. Then using the identity $\sin(\theta) = \sin(\pi - \theta)$ we get the solution $\theta = \pi - .77251 = 2.3691$.

(b) $-\pi \le \theta < \pi$
We still have both the solutions from before, $\theta = .77251$ and $\theta = 2.3691$. Adding or subtracting 2π from either of these solutions will move the angle out of the range $[-\pi, \pi]$.

(c) $-\pi/2 \le \theta < 3\pi/2$
We still have both the solutions from before, $\theta = .77251$ and $\theta = 2.3691$. Adding or subtracting 2π from either of these solutions will move the angle out of the range $[-\pi/2, 3\pi/2]$.

10. Simplify the expression $\frac{\sin\theta}{1+\cos\theta} + \cot\theta$ by adding the two quantities and then manipulating the result.
We have

$$\begin{aligned}\frac{\sin\theta}{1+\cos\theta} + \cot\theta &= \frac{\sin\theta}{1+\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} + \frac{\sin\theta(1+\cos\theta)}{\sin\theta(1+\cos\theta)} \\ &= \frac{\sin^2\theta + \cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)} = \frac{1+\cos\theta}{\sin\theta(1+\cos\theta)} = \frac{1}{\sin\theta} = \csc\theta\end{aligned}$$

11. Verify that $\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} = \frac{2}{\cos^2\theta}$.

$$\begin{aligned}\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} &= \frac{1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)} + \frac{1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)} = \frac{1-\sin\theta}{1-\sin^2\theta} + \frac{1+\sin\theta}{1-\sin^2\theta} \\ &= \frac{1-\sin\theta}{\cos^2\theta} + \frac{1+\sin\theta}{\cos^2\theta} = \frac{1-\sin\theta+1+\sin\theta}{\cos^2\theta} = \frac{2}{\cos^2\theta}\end{aligned}$$

12. Starting with the identities $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ and $\sin(\theta+\phi) = \cos\theta\sin\theta + \cos\phi\sin\phi$, derive a formula for $\tan(\theta+\phi)$.

$$\begin{aligned}\tan(\phi+\theta) &= \frac{\sin(\phi+\theta)}{\cos(\phi+\theta)} = \frac{\sin\phi\cos\theta + \sin\theta\cos\phi}{\cos\phi\cos\theta - \sin\phi\sin\theta} \\ &= \frac{(\cos\phi\cos\theta)^{-1}(\sin\phi\cos\theta + \sin\theta\cos\phi)}{(\cos\phi\cos\theta)^{-1}(\cos\phi\cos\theta - \sin\phi\sin\theta)} = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta}\end{aligned}$$