

Solutions to the Practice Problems
Math 1060
September 19, 2002

1. If $\tan \theta = 3/4$ and $\cos \theta < 0$ find

(a) $\cos \theta$

Recall that $\tan \theta = \frac{\sin \theta}{\cos \theta}$, or

$$\sin \theta = (\cos \theta) \cdot (\tan \theta) = \frac{3}{4} \cos \theta.$$

We also know that

$$\cos^2 \theta + \sin^2 \theta = 1.$$

Plugging the second relation into the first we find

$$1 = \cos^2 \theta + (3/4)^2 \cos^2 \theta = (1 + \frac{9}{16}) \cos^2 \theta = \frac{25}{16} \cos^2 \theta.$$

Therefore $\cos \theta = \pm \sqrt{16/25} = \pm 4/5$. But we know $\cos \theta < 0$, so we know $\cos \theta = -4/5$.

(b) $\sin \theta$

$$\sin \theta = \frac{3}{4} \cos \theta = (\frac{3}{4})(-\frac{4}{5}) = -\frac{3}{5}.$$

(c) $\sec \theta$

$$\sec \theta = \frac{1}{\cos \theta} = -\frac{5}{4}.$$

2. If $\cot \theta = -1/2$ and $\sin \theta > 0$ find

(a) $\sin \theta$

Here we have $-1/2 = \cot \theta = \frac{\cos \theta}{\sin \theta}$, or

$$\cos \theta = -\frac{1}{2} \sin \theta.$$

Again, we plug this into $1 = \cos^2 \theta + \sin^2 \theta$ to obtain

$$1 = (-\frac{1}{2})^2 \sin^2 \theta + \sin^2 \theta = (\frac{1}{4} + 1) \sin^2 \theta = \frac{5}{4} \sin^2 \theta.$$

Thus $\sin \theta = \pm \frac{2}{\sqrt{5}}$. But we also know $\sin \theta > 0$, so $\sin \theta = \frac{2}{\sqrt{5}}$.

(b) $\cos \theta$

$$\cos \theta = (\sin \theta) \cdot (\cot \theta) = (\frac{2}{\sqrt{5}}) \cdot (-\frac{1}{2}) = -\frac{1}{\sqrt{5}}.$$

(c) $\tan \theta$

$$\tan \theta = \frac{1}{\cot \theta} = -2.$$

(d) $\csc \theta$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\sqrt{5}}{2}.$$

3. Suppose $\tan \theta = \sqrt{3}$. Find θ if

(a) $\pi/2 \leq \theta < 3\pi/2$

We know one solution, namely $\theta_0 = \pi/3$. Now we can use the periodicity of \tan to find another solution $\theta = \theta_0 + \pi = \pi/3 + \pi = 4\pi/3$. This choice of θ lies in the correct range (between $\pi/2$ and $3\pi/2$).

(b) $-3\pi/2 \leq \theta < -5\pi/2$

This time we take $\theta_0 = \pi/3$ and subtract 2π , to obtain $\theta = \theta_0 - 2\pi = \pi/3 - 2\pi = -5\pi/3$.

(c) $\pi \leq \theta < 2\pi$

This time we take $\theta_0 = \pi/3$ and add π to obtain $\theta = \theta_0 + \pi = \pi/3 + \pi = 4\pi/3$. Notice this answer coincides with the answer for part (a).

4. Find all the solutions θ to the following equations with $0 \leq \theta < 2\pi$.

(a) $\sin \theta = 1/\sqrt{2}$

We know one solution: $\theta = \pi/4$. To see the other solution, draw a horizontal line through a unit circle and above the x -axis. Notice the line hits the circle twice: once at the angle θ with $0 < \theta < \pi/2$, and once at the angle $\pi - \theta$. (The two triangles you obtain by dropping vertical line segments from the intersection points to the x -axis are congruent.) Thus the other solution is $\theta = 3\pi/4$.

(b) $\tan \theta = -\sqrt{3}$

We know one solution, namely $\theta = 2\pi/3$. We also know $\tan \theta$ is periodic with period π , which tells us another solution: $\theta = 5\pi/3$. Finally, we know from looking at the graph of \tan that it is one-to-one on its period (because it's a strictly increasing function there). Therefore we have found the only two solutions.

(c) $\sin \theta = \cos \theta$

Solution # 1: Because we know $\cos \theta$ and $\sin \theta$ are never 0 at the same value of θ , we can divide this equation by $\cos \theta$ to get $\tan \theta = 1$. We know one solution: $\theta = \pi/4$. Periodicity tells us another solution: $\theta = \pi + \pi/4 = 5\pi/4$. By the same reasoning as the last part ($\tan \theta$ is strictly increasing on its period) we know these are the only two solutions.

Solution # 2: Alternatively, we can use the fact that

$$1 = \sin^2 \theta + \cos^2 \theta = 2 \cos^2 \theta.$$

This equation says $\cos \theta = \sin \theta = \pm 1/\sqrt{2}$, which can only happen at $\theta = \pi/4$ and $\theta = 5\pi/4$.

(d) $\sin^2 \theta = 1 + 2 \cos \theta$

In this case, we can rearrange $\cos^2 \theta + \sin^2 \theta = 1$ to read

$$\sin^2 \theta = 1 - \cos^2 \theta.$$

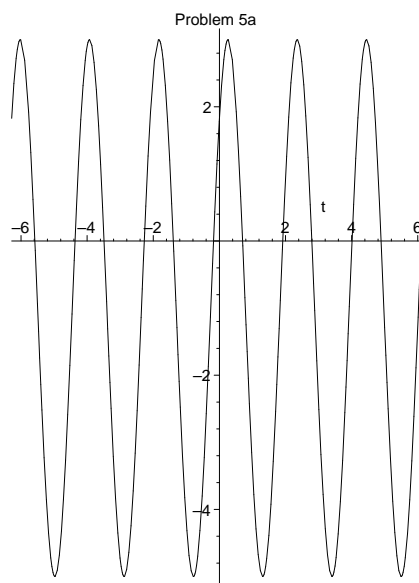
Therefore, our equation really says

$$1 - \cos^2 \theta = 1 + 2 \cos \theta \quad \text{or} \quad -\cos^2 \theta = 2 \cos \theta.$$

We have two possibilities here: either $\cos \theta = 0$ or $\cos \theta = -2$, which is impossible. Therefore the only solutions to our equation occur at $\theta = \pi/2, 3\pi/2$ (where $\cos \theta = 0$).

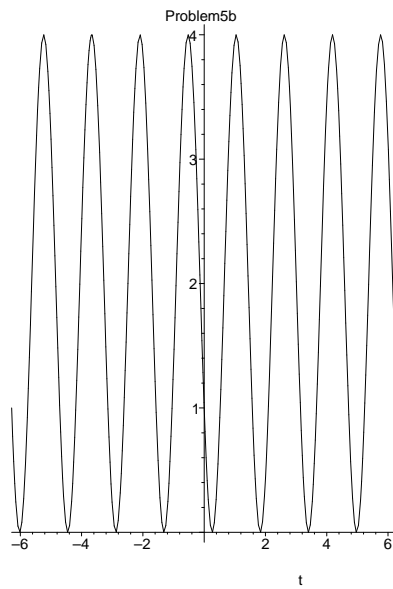
5. Sketch graphs of the following functions. From each sketch, read off the amplitude, phase shift, period and displacement.

(a) $f(t) = 4 \sin(3t + \pi/4) - 1$



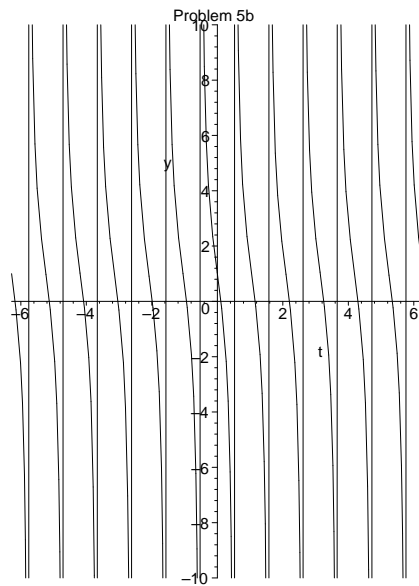
In this case, we can read off that the amplitude is 4, the displacement is -1 , the period is $2\pi/3$ and the phase shift is $(-\pi/4) \cdot (1/3) = -\pi/12$.

(b) $f(t) = -2 \cos(4t - \pi/3) + 2$



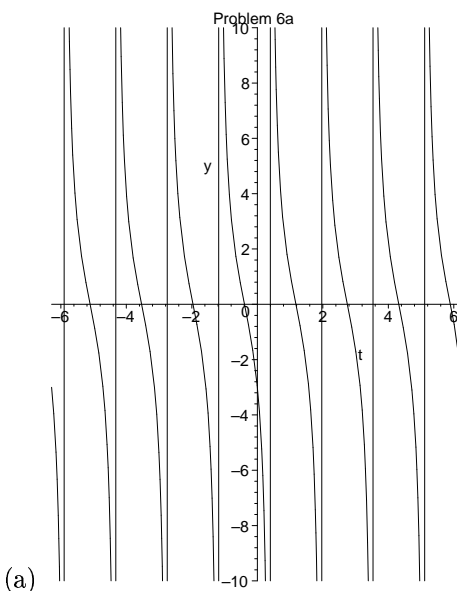
Again, we can read off that the amplitude is 2, the displacement is 2, the period is $2\pi/4 = \pi/2$ and the phase shift is $(\pi/3) \cdot (1/4) = \pi/12$.

(c) $f(t) = 3 \tan(-3t) + 1$

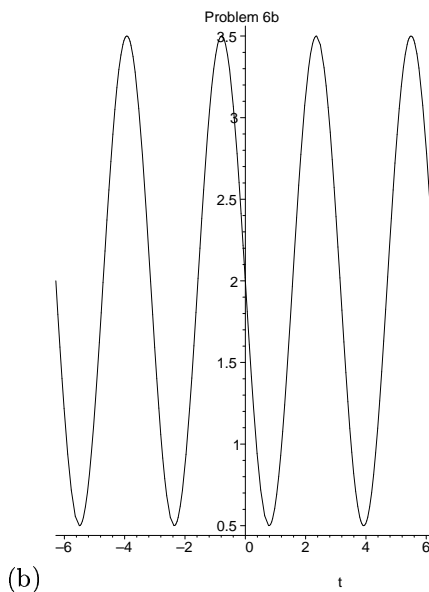


Again, we can read off that the amplitude is 3, the displacement is 1, the period is $\pi/3$ and the phase shift is 0.

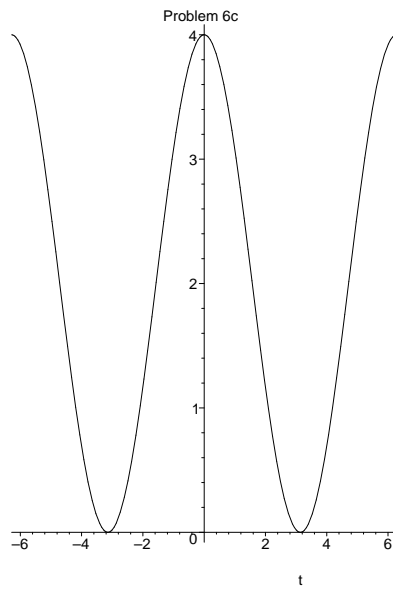
6. Reconstruct each of the following functions from their graphs and read off the amplitude, phase shift, period and displacement.



In this case we can see that this is the graph of a tangent function. The period looks like it is a little bigger than $3/2$, so we'll guess that the coefficient inside the tangent is 2 (to make the period $\pi/2$, which is a little bigger than $3/2$). The tangent also looks to be shifted to the left by $\pi/4$, or half a period. Also, it slopes up to the left instead of the right, so the coefficient multiplying the tangent must be negative. It turns out this is the graph of $-3 \tan(2t + \pi/4)$.

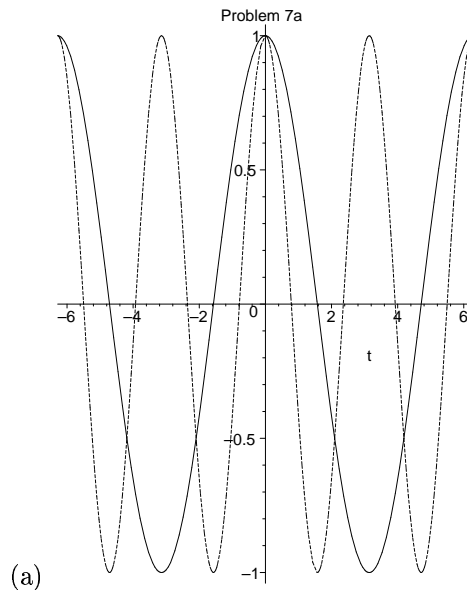


This looks like a $-\sin$ curve, shifted up and scaled vertically and horizontally. The period is π , which is half that of a normal \sin curve, so the horizontal stretching factor is 2. The vertical translation (displacement) is 2 and the amplitude (height of the peaks over the average value) is $3/5$. Putting this all together we see that this is the graph of $-3/2 \sin(2t) + 2$.

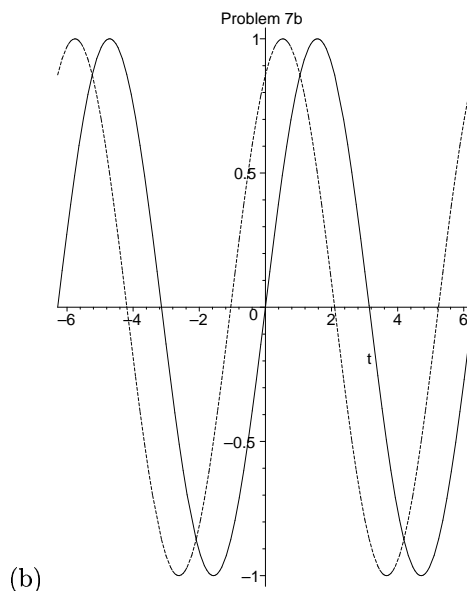


This one looks like a cosine curve, except it is stretched and translated vertically. This function oscillates about 2, so the vertical translation factor is 2. The vertical stretching factor also is 2 because the peak height is 2 above the average value. Therefore we have $2 \cos(t) + 2$.

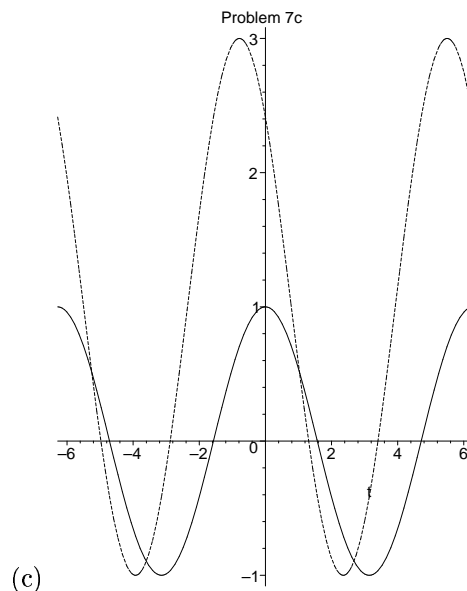
7. In each part below, the graph of f is drawn with a solid line and the graph of g is drawn with a dashed line. For each pair f, g describe how the two functions differ (e.g. by a vertical or horizontal translation) and write down a functional relation for f and g (e.g. one of the functional relations is $g(x) = f(2x)$).



Here we can tell that the phase shift, displacement and amplitude (horizontal translation, vertical translation and vertical scaling, respectively) are all the same. However, the period of g is twice that of f . This means $g(x) = f(2x)$.



Here the displacement, amplitude and period (vertical translation, vertical scaling and horizontal scaling, respectively) are all the same. However, the dotted graph is translated to the left by $\pi/3$ from the solid graph. Thus we have $g(x) = f(x + \pi/3)$.



This one has more going on than the previous two. First, the dotted graph is vertically scaled by a factor of 2 compared to the solid graph (compare the vertical distance between peaks and troughs in the two graphs). Thus the amplitude of g is twice that of f . Second, the dotted graph oscillates about 1 while the solid graph oscillates about 0. Thus the displacement is 1. The period (i.e. horizontal scaling) seems to be the same in both cases, but the horizontal positions of the peaks are not. The peaks are offset by $\pi/4$ (a little bit less than 1) to the left. Putting this all together, we have $g(x) = 2f(x + \pi/4) + 1$.

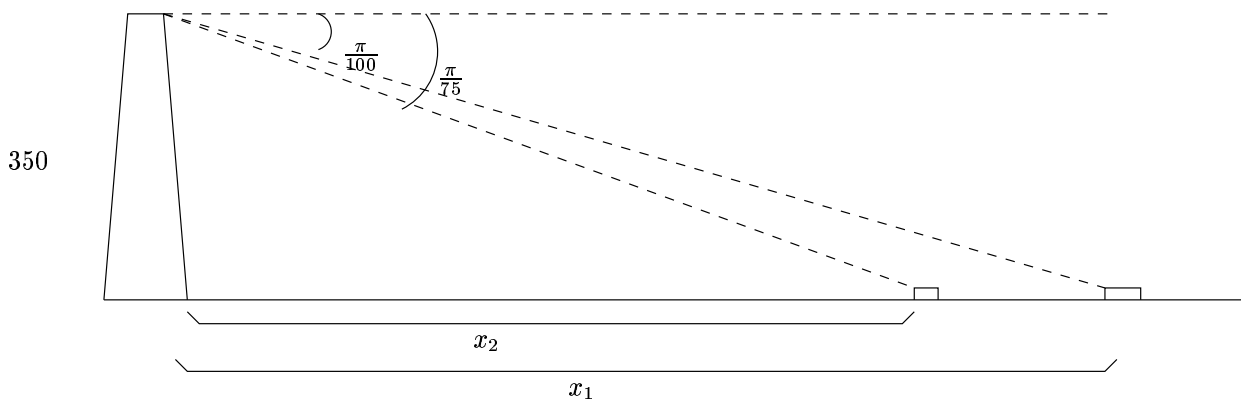
8. Suppose an observer in a lighthouse 350 feet above sea level sees two ships directly offshore. (You can take the shoreline to be a straight line and the ships to be arranged in a straight line perpendicular to the shoreline.) If the angles of depression to the ships are $\pi/100$ and $\pi/75$, how far apart are the ships?

If we call x_1 the distance to the first ship and x_2 the distance to the second ship, then we have (see the figure below)

$$\tan\left(\frac{\pi}{2} - \frac{\pi}{100}\right) = \frac{x_1}{350} \quad \tan\left(\frac{\pi}{2} - \frac{\pi}{75}\right) = \frac{x_2}{350}.$$

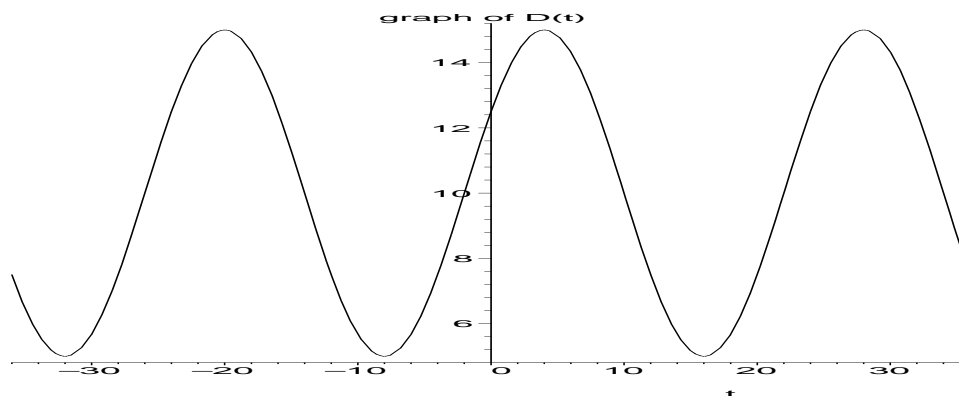
We can use these two equations to write the difference $x_1 - x_2$ as

$$x_1 - x_2 = 350\left(\tan\left(\frac{\pi}{2} - \frac{\pi}{100}\right) - \tan\left(\frac{\pi}{2} - \frac{\pi}{75}\right)\right) = 350\left(\tan\left(\frac{49\pi}{100}\right) - \tan\left(\frac{73\pi}{150}\right)\right) = 2786.43.$$



9. Suppose you want to model the depth of the water at the end of a certain dock using a trigonometric function. Measurements show that the largest depth occurs at 4AM and it is 15 feet. Moreover, the average depth during a given day is 10 feet.

(a) Sketch a graph of $D(t)$.



- (b) Find a function to model the depth of the water of the form $D(t) = a \cos(b(t - c)) + d$, where $D(t)$ is the depth at time t and t measures hours after midnight.

We can construct D either from the graph or from the information given. First, we know that the average value is 10, which tells you $d = 10$. Next, you know that the difference between the peak depth and the average depth is $15 - 10 = 5$, so $a = 5$. Next, we know that the period is 24 (because the tide varies through one full cycle per day), so $b = 2\pi/24 = \pi/12$. Finally, we know that the time of the peak depth (which is the phase shift) is $t = 4$, so $c = 4$. Putting this all together we see that

$$D(t) = 5 \cos\left(\frac{\pi}{12}(t - 4)\right) + 10.$$

- (c) Find a function to model the depth of the water of the form $D(t) = \hat{a} \sin(\hat{b}(t - \hat{c})) + \hat{d}$, where $D(t)$ is the depth at time t and t measures hours after midnight.

There are at least two ways to do this problem. First, you could use the formula $\cos(t - \pi/2) = \sin(t)$ to convert the formula from the previous part to a sin curve. But let's actually go through constructing the function as we did in the previous part. We know the average depth is 10, so $\hat{d} = 10$. Similarly, the difference between the peak depth and the average depth is $15 - 10 = 5$, so $\hat{a} = 5$. Also, we know that the period is 24 (because the tide varies through one full cycle per day), so $\hat{b} = 2\pi/24 = \pi/12$. Finally, we know that the first peak occurs at $t = 4$, which must happen when whatever is inside the sin function is $\pi/2$. Therefore

$$\frac{\pi}{2} = \frac{\pi}{12}(4 - \hat{c}) \quad \text{or} \quad \hat{c} = -2.$$

Putting this all together we see that

$$D(t) = 5 \sin\left(\frac{\pi}{12}(t + 2)\right) + 10.$$

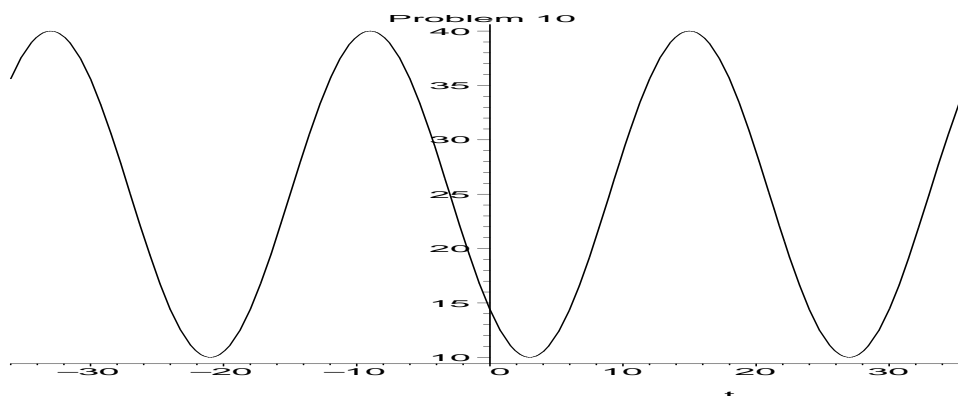
- (d) Suppose a boat requires a depth of at least 8 feet to safely moor to the dock. During what hours can this boat safely moor?

We have to find when the depth is precisely 8 feet; between these two times it is safe to moor. Using the formula from part (b) (one could just as well use the formula from part (c)), we have

$$8 = 5 \cos\left(\frac{\pi}{12}(t - 4)\right) + 10 \quad \text{or} \quad -\frac{2}{5} = \cos\left(\frac{\pi}{12}(t - 4)\right).$$

Taking the arccos of both sides we see $\pm 1.982 = \frac{\pi}{12}(t - 4)$, which gives us $t = \pm 7.572 + 4$. Looking at the graph, we see that the depth larger than 8 when $-3.572 = -7.572 + 4 \leq t \leq 11.572 = 7.572 + 4$. This corresponds to the time period lasting from about 8:30PM to about 11:30AM the next morning.

10. Suppose the average daily temperature during the month of January is given by the following graph.



- (a) What is the average temperature during the day?

The average temperature during the day is the center value about which this trig function oscillates. That center value is 25 degrees.

- (b) What is the peak temperature and when does it occur?

The peak temperature is the highest value of the graph, which is 40 degrees. It occurs when $t = 15$, or at 3PM.

- (c) Find a function to model the temperature of the form $T(t) = a \cos(bt + c) + d$.

We want to find the correct values for a, b, c, d . First, d is the displacement of the cosine curve from an average value of 0, which is nothing more than the average temperature during the day. Thus $d = 25$. The amplitude a tells us how far the peak value is above the average value, which is $40 - 25 = 15$; so $a = 15$. We know that the period has to be 24, because one full day is precisely one period. Thus we have $2\pi/b = 24$, or $b = 2\pi/24 = \pi/12$. Finally, we know that the time of the peak temperature (which is the phase shift) is $t = 15$, and so $c = 15$. Putting this all together, we see

$$T(t) = 15 \cos\left(\frac{\pi}{12}(t - 15)\right) + 25.$$

- (d) How many hours per day is the temperature greater than (or equal to) 30 degrees?

We have to find the times when the temperature is precisely 30 degrees; between those two times the temperature is greater than or equal to 30 degrees. Using the formula we just found for $T(t)$, we see that 30 degrees occurs when

$$30 = 15 \cos\left(\frac{\pi}{12}(t - 15)\right) = 25 \quad \text{or} \quad \frac{1}{3} = \cos\left(\frac{\pi}{12}(t - 15)\right).$$

Taking the arccos of both sides we obtain

$$\pm 1.231 = \frac{\pi}{12}(t - 15) \quad \text{or} \quad t = \pm 4.702 + 15.$$

Thus we see that the temperature is greater than or equal to 30 degrees for $10.298 \leq t \leq 19.702$, which corresponds to a time period lasting from about 10:15AM to about 7:45PM.