Solutions to the Midterm Exam, Math 1060
November 11, 2002

1. Consider the function $F(t) = 2\cos(2\pi(t-1)) + 1$.

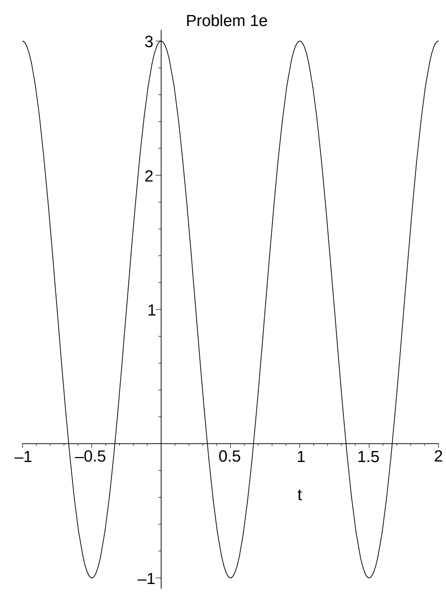
   (a) (3 points) What is the period of F?
   Recall that the factor of 2π acts as a horizontal compression on the graph, changing the period of the cosine curve from 2π to $\frac{2\pi}{2\pi} = 1$.

   (b) (3 points) What is the average value of F?
   Recall that adding 1 to a function acts to translate the graph vertically by one unit. Therefore, the average value of F is 1 plus the average value of a standard cosine curve, which is 0. Thus the average value of F is $1 + 0 = 1$.

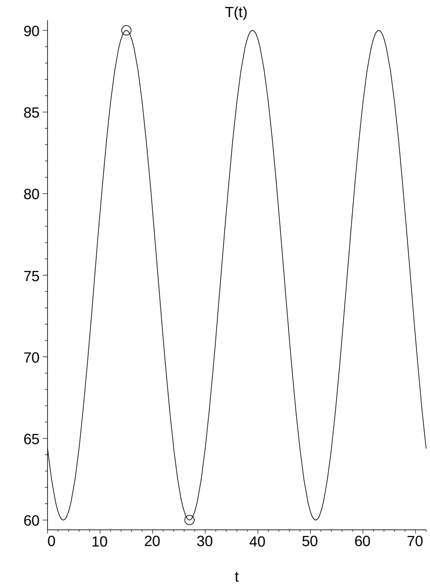
   (c) (3 points) What is the amplitude of F?
   Recall that the factor of 2 multiplying the cosine acts as a vertical scaling, so the new amplitude is twice that of a standard cosine curve, or 2.

   (d) (3 points) What is the phase shift of F?
   Recall that subtracting 1 from the argument of the function (i.e. whatever is inside the parentheses) acts to translate the graph to the right by one unit. This is precisely the phase shift, so the phase shift of F is 1.

   (e) (3 points) Sketch a graph of three periods of F. You might want to label your axes and some points on your graph.



2. Suppose the temperature $T(t)$ during a May day is given by the graph below. Here $T(t)$ is the temperature at time t, where t is the number of hours after midnight. You can assume T has the form $T(t) = a\cos(b(t-c)) + d$ for some numbers a, b, c and d. The two marked points on the graph are $(15, 90)$ and $(27, 60)$.



(a) (3 points) What is the maximum temperature and when does it occur during the day?
Notice that the first marked point, $(15, 90)$ is one of the peaks of the graph. Therefore, the highest temperature is 90 and it occurs 15 hours after midnight, or at 3PM.

(b) (4 points) What is the average temperature and when does it occur during the day?
The other marked point, $(27, 60)$ is a minimum, and it occurs half a period after the maximum. First, we can conclude that the average temperature is the average of 90 and 60, which is 75. Second, we know that the average temperature occurs a quarter period before and after the maximum temperature. A quarter of a period is half of a half-period, which is $\frac{1}{2}(27 - 15) = 6$. Therefore the average temperature occurs at $t = 15 - 6 = 9$, or 9AM, and at $t = 15 + 6 = 21$, or 9PM.

(c) (4 points) Find a formula for $T(t)$ of the form $T(t) = a\cos(b(t - c)) + d$.
We know that the average value of T is 75, so $d = 75$. The size of the oscillations is $90 - 75 = 15$, so $a = 15$. The period is $24 = \frac{2\pi}{b}$, so $b = \frac{\pi}{12}$. Finally, the peak temperature occurs at $t = 15$. Thus, because a standard cosine curve has its peak at 0, we have $0 = \frac{\pi}{12}(15 - c)$, and so $c = 15$. Putting this all together, we see that

$$T(t) = 15\cos(\frac{\pi}{12}(t - 15)) + 75.$$

(d) (4 points) How many hours per day is the temperature at least 95 degrees? (Think before you compute! And be sure to explain your answer.)
None. The maximum temperature is 90, which is strictly less than 95. Therefore the temperature never reaches 95 degrees or higher.

3. Find all the solutions θ to the equation $\tan(3\theta - \pi/4) = 0$ when

(a) (5 points) $\pi/2 \le \theta < 3\pi/2$
We can obtain one solution using the arctangent button on a calculator, namely $3\theta - \pi/4 = 0$. Using the fact that tangent is periodic with period π, we also have

$$3\theta - \pi/4 = 0 + \pi k \qquad \text{or} \qquad \theta = \frac{\pi/4 + \pi k}{3} = \frac{\pi(1 + 4k)}{12}.$$

Here k can be any integer. If we want $\pi/2 \le \theta < 3\pi/2$, we have to choose $k = 2, 3, 4$. One can find these values of k by just testing some numbers until they work. A more systematic way to find which values of k work is the following: the inequalities for θ tell us

$$\frac{\pi}{2} \le \frac{\pi(1 + 4k)}{12} < \frac{3\pi}{2},$$

which we can rearrange to read $5/4 \le k < 17/4$. The only integers in this range are $k = 2, 3, 4$. At any rate, the solutions are $\theta = 3\pi/4, 13\pi/12, 17\pi/12$.

(b) (5 points) $-\pi \le \theta < 0$

Again, we have $\theta = \frac{\pi(1+4k)}{12}$. It remains to find which values of k work. The inequalities on θ tell us

$$-\pi \le \frac{\pi(1+4k)}{12} < 0,$$

which we can rearrange to read $-13/4 \le k < -1/4$. Thus the only choices for k are $k = -1, -2, -3$, and so $\theta = -\pi/4, -7\pi/12, -11\pi/12$.

4. (5 points) Show that

$$\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = \frac{2}{\sin^2\theta}.$$

$$\begin{aligned}\frac{1}{1+\cos\theta} + \frac{1}{1-\cos\theta} &= \frac{1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)} + \frac{1+\cos\theta}{(1+\cos\theta)(1-\cos\theta)} = \frac{1-\cos\theta}{1-\cos^2\theta} + \frac{1+\cos\theta}{1-\cos^2\theta} \\ &= \frac{1-\cos\theta+1+\cos\theta}{1-\cos^2\theta} = \frac{2}{1-\cos^2\theta} = \frac{2}{\sin^2\theta}\end{aligned}$$

5. (5 points) Using the identity

$$\cos(\phi+\theta) = \cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta),$$

express $\cos(2\theta)$ entirely in terms of $\cos(\theta)$.

$$\begin{aligned}\cos(2\theta) &= \cos(\theta+\theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) = \cos^2(\theta) - \sin^2(\theta) \\ &= \cos^2(\theta) - (1-\cos^2(\theta)) = 2\cos^2(\theta) - 1\end{aligned}$$