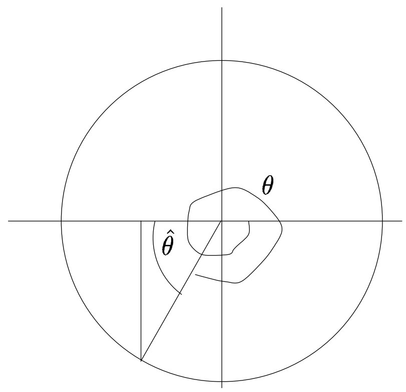
Solutions to the Midterm Exam, Math 1060
October 3, 2002

1. Let $\theta = -\frac{13\pi}{5}$.

   (a) Find an angle $\bar{\theta}$ which is coterminal to θ and such that $0 \leq \bar{\theta} < 2\pi$.
   Remember that all coterminal angles differ by integer multiples of 2π. Adding 2π to θ twice, we get
   $$\bar{\theta} = \theta + 2\pi + 2\pi = -\frac{13\pi}{5} + 4\pi = \frac{7\pi}{5},$$
   which lies in the correct range of angles.

   (b) Find the reference angle $\hat{\theta}$ for θ.



   Draw the angle θ on the unit circle and you'll see that it wraps in the clockwise direction one full rotation and then ends up in the third quadrant. Drop a vertical line segment from a point on the terminal ray of θ to the x-axis; the angle $\hat{\theta}$ is the angle in this right triangle closest to the origin. From this picture we see that
   $$\theta - \hat{\theta} = -3\pi, \quad \text{or} \quad \hat{\theta} = \frac{2\pi}{5},$$
   because rotating first by θ and then by $-\hat{\theta}$ yields one and a half rotations in the clockwise (or negative) direction

2. If $\tan\theta = -4/3$ and $\cos\theta < 0$ find

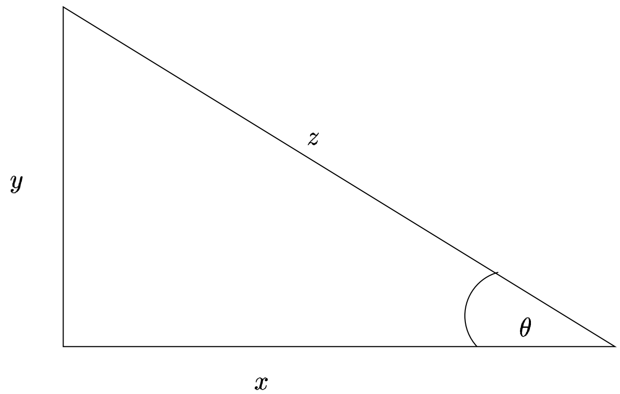
   (a) $\cos\theta$
   We have that
   $$-\frac{4}{3} = \tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \text{so} \quad \sin\theta = -\frac{4}{3}\cos\theta.$$
   Plugging this into $\sin^2\theta + \cos^2\theta = 1$ we see
   $$1 = (-\frac{4}{3}\cos\theta)^2 + \cos^2\theta = (1 + \frac{9}{16})\cos^2\theta = \frac{25}{9}\cos^2\theta,$$
   which tells us $\cos^2\theta = 9/25$. We know to take the negative square root, because we're given $\cos\theta < 0$. Therefore, $\cos\theta = -3/5$.

   (b) $\sin\theta = -\frac{4}{3}\cos\theta = (-\frac{4}{3}) \cdot (-\frac{3}{5}) = \frac{4}{5}$

3. Suppose $\sin\theta = -1/\sqrt{2}$.
   The easiest way to do this problem is the following: you know that $\sin(\pi/4) = 1/\sqrt{2}$ (from the front), so $\sin(-\pi/4) = -1/\sqrt{2}$. Then you also know $\sin(5\pi/4) = -1/\sqrt{2}$ (from the fact that $\sin(\theta) = \sin(\pi - \theta)$; to see this last identity, think of drawing a horizontal line which hits the unit circle twice. If you call the first angle θ the the other angle has to be $\pi - \theta$ because the two have to add up to give you a half a rotation.). Finally, add or subtract the appropriate multiple of 2π to $-\pi/4$ or $5\pi/4$ to obtain an angle in the correct range. Anyhow, here are the answers.
   Find θ if

   (a) $\pi/2 \leq \theta < 3\pi/2$: $\theta = \frac{5\pi}{4}$
   (b) $5\pi/2 \leq \theta < 7\pi/2$: $\theta = \frac{13\pi}{4}$
   (c) $-3\pi/2 \leq \theta < -\pi/2$: $\theta = -\frac{3\pi}{4}$

4. Consider the following right triangle.



(a) If $x = 3$ and $y = 4$ find z.
x, y and z are the sides of a right triangle, with z being the hypotenuse. So we have $z^2 = x^2 + y^2 = 9 + 16 = 25$. Thus $z = 5$.

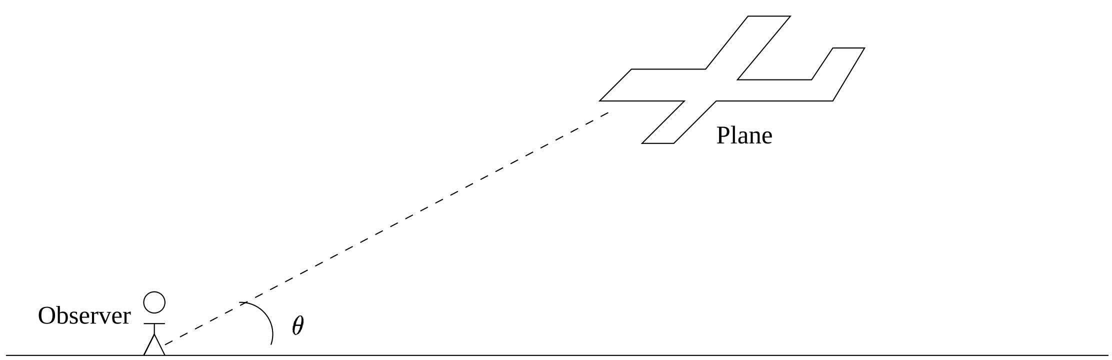
(b) If $x = 2$ and $\theta = \frac{\pi}{5}$ find z.
We have $\cos\theta = \frac{x}{z}$, or $z = \frac{x}{\cos\theta}$. Plugging in the numbers we see

$$z = \frac{2}{\cos(\pi/5)} \cong 2.472.$$

(c) If $y = 1$ and $\theta = \frac{\pi}{8}$ find x.
We have that $\tan\theta = \frac{y}{x}$, or $x = \frac{y}{\tan\theta}$. Plugging in the numbers we see

$$x = \frac{1}{\tan(\pi/8)} \cong 2.414.$$

5. Suppose a plane is flying at a height of 25,000 feet. Label by θ the angle between the ground and the line joining an observer (in fixed position) on the ground and the plane (see the sketch below).



This sketch is not drawn to scale.

For this problem, we will label the distance from the plane to the observer z, the height of the plane y (which is always 25,000 feet) and the horizontal distance from plane to the observer x. This is just a notational convenience.

(a) If $\theta = \pi/6$ find the distance between the plane and the observer.
We have $\sin\theta = \frac{y}{z}$, or $z = \frac{y}{\sin\theta}$. Plugging in the numbers we have

$$z = \frac{25,000}{\sin(\pi/6)} = 50,000 \text{ feet}.$$

(b) If the horizontal distance (i.e. the distance between the shadow of the plane and the observer, assuming the sun is shining straight down) between the plane and the observer is 25,000 feet, find θ. (You can assume $0 \leq \theta \leq \pi/2$.)

Solution #1: This time we have $\tan\theta = \frac{x}{y}$. Plugging in the numbers we see that $\tan\theta = 1$. For $0 \leq \theta \leq \pi/2$ this only happens at $\theta = \pi/4$ (consult this list of angles from the front of the exam).

Solution #2: This time, look at the right triangle with side lengths x, y and z. We know that $x = y$, so this is an isosceles right triangle, so the angle θ has to be $\pi/4$.