MATH 227Q. Fall 2002.
Prof. Olshevsky's section

Key to the solutions to the exam 2.

1. Let

$$A = \begin{bmatrix} 1 & 3 & 4 & 0 \\ -3 & -6 & -7 & 2 \\ 3 & 3 & 0 & -4 \\ -5 & -3 & 2 & 9 \end{bmatrix}.$$

a. Construct an LU factorization for A (Write down the L and U factors).

b. Use the LU factorization to solve a linear system $Ax = b$ with

$$b = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}.$$

Solution.

a. To construct the LU factorization let us first run the Gaussian elimination algorithm.

$$\begin{bmatrix} 1 & 3 & 4 & 0 \\ -3 & -6 & -7 & 2 \\ 3 & 3 & 0 & -4 \\ -5 & -3 & 2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & -6 & -12 & -4 \\ 0 & 12 & 22 & 9 \end{bmatrix} c \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

From here it follows that $A = LU$ with

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -5 & 4 & -1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 3 & 5 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

b. To solve

$$LUx = b$$

we need to solve two triangular systems

$$Ly = b, \qquad Ux = y$$

L is lower triangular and hence the corresponding linear system can be solved by forward substitution yielding

$$y = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

U is upper triangular yielding

$$x = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

2. Find the inverse for the following matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1- & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -3 & -3 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \sim$$

3. Let H and F be the two sets below. Explain which one is a subspace of $\mathbb{R}^3$ and which one is not . Provide ALL details (i.e., a proofs in both cases).

$$H = \left\{ \begin{bmatrix} 3a+b \\ 4 \\ 9a-7c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}, \qquad F = \left\{ \begin{bmatrix} 2a-b \\ 4b+a \\ 9a+7c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\} \tag{1}$$

Solution. H is not a subspace since it is not closed under multiplication by scalars. For example,

$$v = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \in H$$

(a =b = c =), but

$$2v = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}$$

does not belong to H (the second coordinate is not 4 as reqired).

On the other hand, F is a subspace since it is closed under addition and multiplication by scalars. Indeed

- Closed under addition. If $v_1, v_2 \in F$ then

$$v_1 = \begin{bmatrix} 2a_1-b_1 \\ 4b_1+a_1 \\ 9a_1+7c_1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2a_2-b_2 \\ 4b_2+a_2 \\ 9a_2+7c_2 \end{bmatrix}$$

  Hence

$$v_1 + v_2 = \begin{bmatrix} 2(a_1+a_2)-(b_1+b_2) \\ 4(b_1+b_2)+(a_1+a_2) \\ 9(a_1+a_2)+7(c_1+c_2) \end{bmatrix}$$

  i.e., it has the form shown in (1) so it belongs to F.

- Closed under multiplication by scalars. If $v \in F$ then

$$v = \begin{bmatrix} 2a-b \\ 4b+a \\ 9a+7c \end{bmatrix}$$

  Hence

$$\alpha v = \begin{bmatrix} 2(\alpha a)-(\alpha b) \\ 4(\alpha b)+(\alpha a) \\ 9(\alpha a)+7(\alpha c) \end{bmatrix}$$

  i.e., it has the form shown in (1) so it belongs to F.

4. [15] Let $\mathbf{b}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$, and $H = span\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ It may be verified that $4b_1 + 5b_2 - 3b_3 = 0$. Use this information to find a basis for H. You should not only find a basis, but also provide an explanation (a proof) that is a basis for H.

Solution.

Since $4b_1 + 5b_2 - 3b_3 = 0$ hence $\{b_1, b_2, b_3\}$ are linearly dependent by the definition.

There are two procedures to come up with a basis.

(a) The first procedure is to form a matrix $B = \begin{bmatrix} \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{bmatrix}$ and then find a basis in its column space.

(b) The second procedure is easier. Observe that

- Since $\mathbf{b_3}$ is a linear combination of $\mathbf{b_1}, \mathbf{b_2}$ hence

$$H = \text{span}\{b_1, b_2\}$$

- Moreover $\{b_1, b_2\}$ is a linearly independent set (they are not scalar multiples of each other).

Hence $\{b_1, b_2\}$ is a basis for H.

5. By evaluating the determinant of the matrix

$$A = \begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$$

figure out if it is invertible (non-singular) or not.

Solution. Expanding the determinant along the 4-th row we have

$$\det A = -3 \cdot \det \begin{bmatrix} 3 & 2 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ -5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{bmatrix}$$

Now, expanding the remaining determinant down the last column we have:

$$\det A = (-3)(-1) \cdot \det \begin{bmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Expanding the remaining determinant down the first column we have:

$$\det A = (-3)(-1) \cdot (3 \det \begin{bmatrix} -4 & 1 \\ 3 & 2 \end{bmatrix} + 2 \det \begin{bmatrix} 2 & 4 \\ -4 & 1 \end{bmatrix})$$

So,

$$\det A = (-3)(-1) \cdot (3(-11) + 2(18)) = 9$$

6. Assume that the following two matrices are row equivalent:

$$A = \begin{bmatrix} -3 & -2 & 3 & 8 & 7 \\ 1 & 1 & -2 & -2 & 3 \\ -4 & -3 & 5 & 10 & 4 \\ 2 & 1 & -1 & -5 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 7 \\ 0 & 1 & -3 & 0 & 6 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the basis for the column space of A and the basis for the nullspace of A. State the rank of A and the dimension of the nullspace of A.

Solution. The reduced matrix B has pivots in the 1st, 2nd and the 4th columns The basis for $ColA$ consists of the 1st, 2nd and the 4th columns of A:

$$\begin{bmatrix} -3 \\ 1 \\ -4 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 8 \\ -2 \\ 10 \\ -5 \end{bmatrix},$$

The dimension of $ColA$ is 3 and hence (by definition) the rank of A is 3. Since A is 5×5 hence Theorem 14 implies that the dimension of the nullspace is 2.

It remains to find the basis for $NullA$, i.e., to solve the homogenuous system $A\mathbf{x} = \mathbf{0}$, or, which is the same (indeed A and B are row equivalent) to solve the homogenuous system $B\mathbf{x} = \mathbf{0}$, i.e.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - 7x_5 \\ 3x_3 - 6x_5 \\ x_3 \\ -5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

7. Prove the theorem: *The null space of an $m \times n$ matrix A is a subspace of* $\mathbb{R}^n$.

Solution. See p. 170.