MATH 227Q. Fall 2006.
Prof. Olshevsky's section

Solutions to Exam 1

1. **[20 points]** Find the general solution of the following homogeneous linear system of equations.

$$\begin{array}{ccccccccc} x_1 & - & 3x_2 & - & 9x_3 & + & 5x_4 & = & 0 \\ & & x_2 & + & 2x_3 & - & x_4 & = & 0 \end{array}$$

Solution. To describe the solution set we reduce A to its reduced row echelon form:

$$A = \left[\begin{array}{cccc|c} 1 & -3 & -9 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right].$$

Solve for basic variables x_1 and x_2 with x_3 and x_4 free:

$$\begin{array}{lcccl} x_1 & & -3x_3 & +2x_4 & = 0 \\ & x_2 & +2x_3 & -x_4 & = 0 \end{array}, \qquad \begin{array}{lcrr} x_1 & = & 3x_3 & -2x_4 \\ x_2 & = & -2x_3 & +x_4 \end{array}$$

As a vector, the general solution of $A\mathbf{x} = \mathbf{0}$ has the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 - 2x_4 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

2. **[20 points]** Find the general solution of the following non-homogeneous linear system of equations.

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & x_3 & = & 3 \\ x_1 & - & 3x_2 & + & x_3 & = & 4 \\ -5x_1 & & & + & 2x_3 & = & -5 \end{array}$$

Solution. To describe the solution set we reduce A to its reduced row echelon form:

$$A = \left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ 1 & -3 & 1 & 4 \\ -5 & 0 & 2 & -5 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 2 & -1 & 1 & 3 \\ -5 & 0 & 2 & -5 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 0 & 5 & -1 & -5 \\ 0 & -15 & 7 & 15 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 0 & 5 & -1 & -5 \\ 0 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 0 & 5 & -1 & -5 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -3 & 0 & 4 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The solution is unique:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

3. **[20 points]** Determine which of the following sets of vectors are linearly independent. Give reasons for your answers.

(a) $\mathbf{v_1} = \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

(b) $\mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u_3} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$.

(c) $\mathbf{r_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{r_2} = \begin{bmatrix} -2 \\ 1 \\ -3 \\ 5 \end{bmatrix}$, $\mathbf{r_3} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{r_4} = \begin{bmatrix} 0 \\ 9 \\ -4 \\ 4 \end{bmatrix}$.

Solution.

(a) Answer: linearly dependent, since $\mathbf{v_1}$ is a scalar multiple of $\mathbf{v_2}$, i.e., $\mathbf{v_1} = 2\mathbf{v_2}$.

(b) Answer: linearly dependent, since $\mathbf{u_2} = \mathbf{0}$ so that

$$0 \cdot \mathbf{u_1} + \alpha \cdot \mathbf{u_2} + 0 \cdot \mathbf{u_3} = \mathbf{0}$$

with any nonzero α.

(c) Let us form the matrix

$$A = [\ \mathbf{r_1} \mid \mathbf{r_2} \mid \mathbf{r_3} \mid \mathbf{r_4}\ ] = \left[\begin{array}{c|c|c|c} 1 & -2 & 1 & 0 \\ 1 & 1 & -1 & 9 \\ 0 & -3 & 2 & -4 \\ -1 & 5 & 3 & 4 \end{array}\right]$$

On p. 66 of the textbook we have a statement: " *The columns of the matrix A are linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has only trivial solution.*" In other words, The columns of the matrix A are linearly independent if and only if the matrix A has 4 pivots.

It remains to reduce A and to count its pivots.

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & 1 & -1 & 9 \\ 0 & -3 & 2 & -4 \\ -1 & 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 9 \\ 0 & -3 & 2 & -4 \\ 0 & 3 & 4 & 4 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 9 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 6 & -5 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{3} & -2 & 9 \\ 0 & 0 & \boxed{6} & -5 \\ 0 & 0 & 0 & \boxed{5} \end{bmatrix}$$

Since the 4×4 matrix A has 4 pivots, its columns are linearly independent.

4. **[20 points]** Let $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ -1 & -3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix}$, and define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

(a) Determine if $\mathbf{y}$ is in the range of T. (Check your arithmetic).

(b) Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$? Why or why not? Explain.

Solution.

(a) $\mathbf{y}$ is in the range of T if and only if $T(\mathbf{x}) = \mathbf{y}$ for some $\mathbf{x}$, i.e., if and only if the linear system $A\mathbf{x} = \mathbf{y}$, i.e.,

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \\ -1 & -3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix}$$

has a solution. So, let us form the augmented matrix $\begin{bmatrix} A & \mathbf{y} \end{bmatrix}$ and reduce it.

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -2 & -1 & -7 \\ -1 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right].$$

Since the system $A\mathbf{x} = \mathbf{y}$ has a solution (indeed, the row echelon form of the augmented matrix $\begin{bmatrix} A & \mathbf{y} \end{bmatrix}$ does not have a row of the form $\begin{bmatrix} 0 & \cdots & 0 & * \end{bmatrix}$) hence $\mathbf{y}$ is in the range of T.

(b) By Theorem 12, if T would be "onto" if and only if the columns of its matrix A span $\mathbb{R}^3$. The columns of A A span $\mathbb{R}^3$ if and only if $A\mathbf{x} = \mathbf{b}$ is solvable for any $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Let us form the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ and reduce it.

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ -2 & -1 & b_2 \\ -1 & -3 & b_3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 + 2b_1 \\ 0 & -2 & b_3 + b_1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 + 2b_1 \\ 0 & 0 & b_3 + 2b_2 + 5b_1 \end{array}\right].$$

We see that if $b_1 = b_2 = b_3 = 1$ there is a row of the form $\begin{bmatrix} 0 & \cdots & 0 & * \end{bmatrix}$ and the system $A\mathbf{x} = \mathbf{b}$ is inconsistent meaning that T does not map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

5. **[20 points]** Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation such that

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}, \qquad T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}.$$

   (a) Find the matrix of T.

   (b) Find a formula for for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

Solution.

(a) By theorem 10 the matrix of T is $A = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix}$.

(b) Since

$$A \cdot x = \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{bmatrix}$$

the desired formula looks like $T(x_1, x_2) = (5x_1 - 3x_2, -7x_1 + 8x_2, 2x_1)$.

6. **[20 points]** Prove the following theorem. *If a set* $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ *in* $\mathbb{R}^n$ *contains more vectors than there are entries, i.e., if* $k > n$*, then the set is linearly dependent.*

Solution. See p. 69, theorem 8.

Please notice that just giving an example (e.g., with three vectors each having two entries) is not a **PROOF**.