Math 227 Sample Final Exam Answers

1. The eigenvalues are 6, 4, and -1. For $\lambda = 6$, the eigenvectors are of the form $x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (for any $x_1 \neq 0$), so one eigenvector would be $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. For $\lambda = 4$, the eigenvectors are of the form $x_3 \begin{bmatrix} -5/3 \\ 1/3 \\ 1 \end{bmatrix}$ (for $x_3 \neq 0$), so an eigenvector would be $\begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$. Finally, the eigenvectors for $\lambda = -1$ have the form $x_3 \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}$ (for $x_3 \neq 0$), so $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

2. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

3. (a) The eigenvalue for $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is 2, the eigenvalue for $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is 8 and the eigenvalue for $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is 1.

   (b) $P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (c) The eigenvalues of D are the same as the eigenvalues of A, since A and D are similar. The eigenvectors for $\lambda = 2$ are all vectors of the form $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$, the eigenvectors for $\lambda = 8$ are of the form $\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$ and the eigenvectors for $\lambda = 1$ are of the form $\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$. In all cases, a is any non-zero real number.

4. (a) Since $\begin{bmatrix} -1 & 3 & 1 & 20 \\ 1 & 3 & 2 & 16 \\ 4 & 2 & 3 & 4 \end{bmatrix}$ is row-equivalent to $\begin{bmatrix} 1 & 0 & .5 & -2 \\ 0 & 1 & .5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we see that the answer is yes, and that

$$\begin{bmatrix} 20 \\ 16 \\ 4 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} + 6 \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

   (b) No. Since the matrix whose columns are the vectors in $\mathcal{B}$ does not row reduce to the identity (so does not have a pivot position in every row), there are vectors in $\mathbb{R}^3$ which are not linear combinations of the vectors in $\mathcal{B}$.

   (c) No. There is a free variable in the matrix whose columns are the vectors in $\mathcal{B}$. In fact, we see that $.5 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} + .5 \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

5. (a) T is NOT 1-1. This can be seen a variety of ways (the columns are not linearly independent, there is a free variable, or even the fact that there is no 1-1 linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$). However, we can show explicitly that the map is not 1-1 by producing two vectors which get mapped to the same thing. Note that $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is a non-trivial vector in the null space of the matrix, so both it and the zero vector in $\mathbb{R}^3$ get mapped to the zero vector in $\mathbb{R}^2$.

(b) T is onto, since there is a pivot position in every row of the matrix. Thus the columns span $\mathbb{R}^2$, so everything in $\mathbb{R}^2$ is a linear combination of the columns. That is, for every vector $\mathbf{y}$ in $\mathbb{R}^2$, there is a vector $\mathbf{x}$ in $\mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{y}$.

6. (a) Notice first that the vectors $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$ are orthogonal, so we can use our formula for the projection. This gives

$$\text{proj}_W(\mathbf{y}) = \frac{39}{26}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \frac{25}{25}\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \\ 9 \end{bmatrix}.$$

(b) $\mathbf{y} = \begin{bmatrix} 1.5 \\ 0.5 \\ 9 \end{bmatrix} + \begin{bmatrix} 12.5 \\ -1.5 \\ -2 \end{bmatrix}.$

7. (a) A is row equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b)
1. A is invertable.
2. $\det(A) \neq 0$.
3. A has 3 pivot positions.
4. $A\mathbf{x} = 0$ has only the trivial solution.
5. $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^3$.
6. The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.
7. The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is 1-1.
8. $\text{Col}A = \mathbb{R}^3$.
9. $\text{rank}A = 3$.
10. $\text{Nul}A = \{\mathbf{0}\}$.
11. $\dim \text{Nul}A = 0$.
12. $\text{Row}A = \mathbb{R}^3$.
13. The columns of A are linearly independent.
14. The columns of A span $\mathbb{R}^3$.
15. The columns of A form a basis for $\mathbb{R}^3$.
16. A
17. A^T is invertable.
18. The number 0 is not an eigenvalue of A.