Math 227 Sample Exam 2 Answers

1. (a) ...the set of all vectors which are a linear combination of the columns of A.
   (b) ... $\mathcal{B}$ is linearly independent and Span$\mathcal{B} = H$.
   (c) ... the dimension of the column space of A.

2. (a) $A^T = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 5 \end{bmatrix}$
   (b) $\det(A) = 1 \cdot 1 \cdot 4 \cdot 5 = 20$.
   (c) Yes. Since $\det(A) \neq 0$, A is invertible. By the invertible matrix theorem, every vector in $\mathbb{R}^4$ is a linear combination of the columns of A. Thus the columns of A span $\mathbb{R}^4$, and since Col$A$ is a subset of $\mathbb{R}^4$, they are equal. Alternatively, since there are 4 pivot columns, the dimension of the column space (that is, the rank) is 4, and the only 4 dimensional subspace of $\mathbb{R}^4$ is $\mathbb{R}^4$ itself.

3. (a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ -2 \\ 2 \end{bmatrix} \right\}$
   (b) Since the third vector is a linear combination of the first two (in fact, simply the sum of the first two), we can eliminate it without changing the spanned set. The first two vectors are linearly independent (neither is a multiple of the other), so the first two vectors form a basis:
   $$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$
   (c) There are three pivot columns, and 5 columns all together, so the dimension of Nul$B$ is 2.
   (d) We first must put B into reduced echelon form, which is almost done in A, but not quite. We finish the job and get
   $$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
   Then x_3 and x_5 are our free variables, and we see that in parametric vector form, the solutions to $B\mathbf{x} = \mathbf{0}$ are:
   $$\mathbf{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$
   Thus a basis for the null space of B is:
   $$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

4. (a) To verify this, we must find values of a, b, c, and d that $\mathbf{x}$ has the form needed. This amounts to finding the solution to the matrix equation

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 1 & 2 & -2 & 1 \\ 1 & 2 & 10 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -4 \\ 8 \end{bmatrix}$$

The corresponding augmented matrix is row equivalent to

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the system is consistent, we know that there is a solution (in fact, we can let $a = -2$, $b = 0$, $c = 1$, and $d = 0$) so $\mathbf{x}$ is in H.

(b) As stated, the problem is false. First, note that the two vectors in $\mathcal{B}$ are linearly independent, since neither is a multiple of the other. Thus we must simply check that the span of $\mathcal{B}$ is H. To do this, we should check that each of the columns of the matrix we used in part (a) could be expressed as a linear combination of the vectors in $\mathcal{B}$. (If they could, then since H is the column space of that matrix, we would be done.) This fails. I suspect the problem was meant to have

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 10 \end{bmatrix} \right\}$$

as those are the pivot columns of the matrix used in part (a). That is a basis for H, since H is the column space of the matrix in question.

(c) This cannot be done, as $\mathbf{x}$ is not a member of H (nor the space spanned by the $\mathcal{B}$ in part (b)). However, the process we would use is this: form the augmented matrix whose columns are the vectors in $\mathcal{B}$, followed by $\mathbf{x}$. Row reduce. This gives the weights for $\mathbf{x}$ as a linear combination of the vectors in $\mathcal{B}$. The vector (in $\mathbb{R}^2$) whose entries are those weights is $[\mathbf{x}]_{\mathcal{B}}$.