Math 2110-7 **Exam 1** October 2, 2008

(16pts) 1. Provide an example of each of the following, if one exists. If you find an example, briefly state why your example works. If there is no example, explain why that is the case.

(a) A vector $\mathbf{v}$ orthogonal to $\langle 1, -4, 2\rangle$

(b) Vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \times \mathbf{v}$.

(c) A vector function which is **not** continuous.

(d) A function of two variables which has no critical points.

(9pts) 2. Let $\mathbf{a} = \langle 4, 1, -3\rangle$ and $\mathbf{b} = \langle 2, -5, 1\rangle$. Compute the following:

(a) $2\mathbf{a} + \mathbf{b}$

(b) $\mathbf{a} \cdot \mathbf{b}$

(c) $\mathbf{a} \times \mathbf{b}$

(9pts) 3. Suppose the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is $\pi/4$, and that $|\mathbf{u}| = 3$ and $|\mathbf{v}| = 2$. Find:

(a) $\mathbf{u} \cdot \mathbf{v}$

(b) $|\mathbf{u} \times \mathbf{v}|$.

(c) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$

4. Let $f(x, y) = e^{xy} + x^3 - 8$.

(12pts) (a) Find both partial derivatives and all four second partial derivatives of f.

(3pts) (b) Find $\nabla f(1, 0)$, the gradient vector of f at the point (1,0).

(4pts) (c) Find $D_{\langle 1/\sqrt{5}, 2/\sqrt{5}\rangle} f(1, 0)$, the directional derivative of f at $(1, 0)$ in the direction of $\langle 1/\sqrt{5}, 2/\sqrt{5}\rangle$.

(4pts) (d) Is $\langle 1, 2\rangle$ the direction of maximal growth in f at $(1, 0)$? Explain.

(14pts) 5. (a) Find the equation of the plane tangent to the surface $z = x^2 - y^2 + xy$ at the point $P(3, -2, -1)$.

(b) Find a normal vector to the plane you found in part (a).

(c) Find parametric equations for the line which passes through P which is orthogonal to the tangent plane.

(d) At what point does the line you found above intersect the xy-plane?

(14pts) 6. Fredrick the fearless fruit fly zips into the oven just as the door is closed. His position in the oven is given by the vector function $\mathbf{r}(t) = \langle \cos(\pi t), \sin(\pi t), 2\cos(\pi t)\rangle$ (the origin is in the very center of the oven).

(a) Find the position of the fly at $t = 4$

(b) Find a vector function to express the velocity of the fly at time t.

(c) Find the velocity of the fly at $t = 4$.

(d) Suppose the temperature T of the fly is given as a function of its position. Specifically, $T(x, y, z) = \sin(xy) + 2z^2 + 225$. At what rate is the fly's temperature changing, with respect to time, when $t = 4$.

(15pts) 7. Let $f(x, y) = 3y^2 - y\cos(x)$. Determine which of the points below are critical points of f. Then classify the critical points as local maximums, local minimums, or neither (saddle points).

$P\,(0, 1/6)$ $\quad Q(0, 0)$ $\quad S\,(\pi/2, 0)$ $\quad T\,(3\pi/2, 0)$

(5pts) 8. BONUS: Let $W(u,v) = F(x(u,v), y(u,v)) = \int_x^y \frac{\sin(t)}{t}\, dt$, where $x = 3u - \ln(uv)$ and $y = 4u^2v - v^2$.

(a) Find $W(1,1)$.

(b) Find the maximal rate of change in W at the point $(1,1)$ and the direction in which it occurs.

ANSWERS:

1. (a) For instance, $\langle 1, 0, -1/2\rangle$. Any vector $\langle x, y, z\rangle$ such that $\langle x, y, z\rangle \cdot \langle 1, -4, 2\rangle = 0$.
   (b) Not possible. $\mathbf{u} \cdot \mathbf{v}$ is a scalar (number) while $\mathbf{u} \times \mathbf{v}$ is a vector.
   (c) For instance $\langle 1/t, t, 1\rangle$, which is not continuous since one of its components, $1/t$ is not continuous.
   (d) For instance, $f(x,y) = x + y$. In fact, any non-horizontal plane would work, since then the gradient would never be 0.

2. (a) $\langle 10, -3, -5\rangle$
   (b) $8 - 5 - 3 = 0$
   (c) $\mathbf{i}(1 - 15) - \mathbf{j}(4 + 6) + \mathbf{k}(-20 - 2) = \langle -14, -10, -22\rangle$

3. (a) Since $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, we have $6\cos(\pi/4) = 3\sqrt{2}$.
   (b) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta = 3\sqrt{2}$, coincidentally.
   (c) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$, since $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$.

4. (a) $f_x = ye^{xy} + 3x^2$, $f_y = xe^{xy}$. For the second partials, we have $f_{xx} = y^2e^{xy} + 6x$, $f_{yy} = x^2e^{xy}$, and $f_{xy} = f_{yx} = e^{xy} + yxe^{xy}$.
   (b) $\nabla f(x,y) = \langle ye^{xy} + 3x^2, xe^{xy}\rangle$ so $\nabla f(1,0) = \langle 3, 1\rangle$
   (c) $D_{\langle 1/\sqrt{5}, 2/\sqrt{5}\rangle} f(1,0) = \nabla f(1,0) \cdot \langle 1/\sqrt{2}, 2/\sqrt{2}\rangle = 3/\sqrt{5} + 2/\sqrt{5} = \sqrt{5}$.
   (d) No, since the direction of maximal growth of f and $(1,0)$ is $\nabla f(1,0)$, which is not parallel to $\langle 1, 2\rangle$.

5. (a) $z + 1 = 4(x - 3) + 7(y + 2)$.
   (b) $\langle 4, 7, -1\rangle$.
   (c) $x = 3 + 4t$, $y = -2 + 7t$, $z = -1 - t$.
   (d) We must have $z = 0$, so $t = -1$. Thus $x = -1$ and $y = -9$, so $(-1, -9, 0)$.

6. (a) $\mathbf{r}(4) = \langle 1, 0, 2\rangle$
   (b) $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -\pi\sin(\pi t), \pi\cos(\pi t), -2\pi\sin(\pi t)\rangle$
   (c) $\mathbf{v}(4) = \langle 0, \pi, 0\rangle$.

(d) Using the chain rule, we see that $\frac{dT}{dt} = \frac{\partial T}{\partial y}\frac{dy}{dt}$ (the other terms in the sum will be 0). Since $\frac{\partial T}{\partial y} = x\cos(xy)$, we get $\frac{dT}{dt} = 1\cos(0)\cdot\pi = \pi$.

7. We have $f_x = y\sin(x)$, $f_y = 6y - \cos(x)$, $f_{xx} = y\cos x$, $f_{yy} = 6$, and $f_{xy} = \sin(x)$. Thus we see that P, S, and T are critical points, but Q is not (since $f_y(0,0) \neq 0$).

   To classify the critical points, we plug them into $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2$. We find $D(0,1/6) > 0$, $D(\pi/2,0) < 0$ and $D(3\pi/2,0) < 0$. Thus S and T are saddle points, while P is either a local max or a local min. Since $f_{xx}(0,1/6) > 0$, we have, by the second derivative test, that P is a local minimum.

8. (a) First we find that when $(u,v) = (1,1)$, that $(x,y) = (3,3)$. Thus $W(1,1) = \int_3^3 \sin(t)/t dt = 0$.

   (b) The direction of greatest rate of change is the gradient vector. To find it, we must find W_u and W_v. Both of these require the chain rule, and both the observation that $F_x = -\sin(x)/x$ and $F_y = \sin(y)/y$. We find that $\nabla W(1,1) = \rangle 5\sin(3)/3, \sin(3)/3\rangle$. The rate of change in this direction is $|\nabla W(1,1)| = \sqrt{26}\sin(3)/3$.