Practice Problems

Math 210

Sept. 14, 2006

These problems are not in any particular order. The exam will be shorter (about or 5 problems). The last 5 problems compose the first exam for the section of 210 I taught last fall.

1. Given the following pairs of vectors $\vec{u}$ and $\vec{v}$, find the angle θ between them and compute the cross product $\vec{u} \times \vec{v}$.
   (a) $\vec{u} = (1, 2, 3)$, $\vec{v} = (-2, 1, 0)$
   (b) $\vec{u} = (1, 0, 2)$, $\vec{v} = (2, 1, 0)$
   (c) $\vec{u} = (1, 1, 0)$, $\vec{v} = (1, 0, 1)$
2. Consider the plane Π_1 containing $p = (2, 1, 3)$, with normal vector $\vec{n} = (-1, 2, 0)$.
   (a) Write down the linear equation any point (x, y, z) in this plane must satisfy.
   (b) Find the angle between the plane Π_1 and the plane Π_2 determined by $x - y = 2$.
   (c) Parameterize the line l which is the intersection of Π_1 and Π_2.
   (d) Find the distance between the plane Π_1 and the point $q = (3, 3, 3)$.
3. Consider the vectors $\vec{u} = (1, 2, 1)$ and $\vec{v} = (0, 1, -1)$.
   (a) Explain why all the planes parallel to both $\vec{u}$ and $\vec{v}$ will have the same normal vectors (up to scaling).
   (b) Are any of these planes parallel to the plane given by $x + y + z = 2$? Explain your answer.
4. Consider the plane curve given by $c(t) = (\cos(t), \sin(2t))$, for $0 \le t \le 2\pi$.
   (a) Sketch this curve.
   (b) Set up, but do not evaluate, the integral to compute the arclength of c.
   (c) Notice c is periodic ($c(0) = c(2\pi)$). Is c a simple closed curve? In othre words, are the t parameters 0 and 2π the only times c crosses itself?
5. (a) Consider the right circular cone C, with vertex at $(0, 0, 0)$, and slope 1. In other words, the cone C is what you get when you rotate the line $y = z$ in the $y - z$ plane about the z-axis. Write C in cylindrical coordinates.
   (b) Write the part of the shell $1 \le x^2 + y^2 + z^2 \le 4$ lying in the $x < 0, y > 0, z < 0$ octant in spherical coordinates.
6. Consider the space curve $c(t) = (\cos(t), \sin(t), t)$.
   (a) Is the velocity vector ever tangent to the x-axis?
   (b) Verify the Fundamental Theorem of Calculus by checking
   $$c(2\pi) - c(0) = \int_0^{2\pi} c'(t)dt.$$
   (c) Compute the arclength of c for $0 \le t \le 2\pi$.
7. Consider the function $f(x, y) = x^2 - y^2$.
   (a) Sketch the level sets $f = 0$ and $f = 1$.
   (b) Does f have an upper bound? How about a lower bound?
   (c) Compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$.
   (d) Is the tangent plane to the graph of f ever parallel to the $x - y$ plane?
8. Explain why the tangent plane to the graph of a function $f(x, y)$ cannot ever be parallel to the $x - z$ or $y - z$ planes, provided f has continuous partial derivatives.

9. Consider the vectors $\vec{a} = (2, 1, 3)$ and $\vec{b} = (-1, 0, 1)$.

   (a) (3 points) Compute the cross product $\vec{a} \times \vec{b}$.

   (b) (3 points) Find a vector $\vec{x}$ which is perpendicular to $\vec{a}$ and verify that $\vec{x} \perp \vec{a}$. (There are many correct answers.)

   (c) (4 points) Write $\vec{a}$ as a sum $\vec{a} = \vec{u} + \vec{v}$ where $\vec{u}$ is parallel to $\vec{b}$ and $\vec{v}$ is perpendicular to $\vec{b}$. (Hint: you only need to find one of $\vec{u}$ and $\vec{v}$. It might help to draw a picture.)

10. Consider the curve $c(t)$ given by

$$c(t) = (t \cos t, t \sin t, t).$$

   (a) (3 points) Find the velocity and acceleration vectors of this curve.

   (b) (4 points) Is the tangent line to c ever parallel to the $x - y$ plane? Be sure to explain your answer.

   (c) (3 points) Set up, but do not evaluate, the integral to compute the arclength of c for $0 \leq t \leq \pi$.

11. Consider the planes Π_1 and Π_2, given as follows. The first plane Π_1 passes through $p = (1, 2, 3)$ and has the normal vector $\vec{n} = (1, 0, -1)$. The second plane Π_2 is given by the linear equation $x + y + z = 1$.

   (a) Explain how one can tell that Π_1 and Π_2 are not parallel, and compute the cosine of the angle θ between them.

   (b) The two planes Π_1 and Π_2 intersect in a line l. Find a parameterization for l.

12. Consider the function $f(x, y) = x^2 + 4y^2$.

   (a) Sketch the $f = 4$ level set.

   (b) For which values of z does the level set $f = z$ not contain any points? Be sure to explain your answer.

13. Consider the function $f(x, y) = x^2 y + y^2 x$.

   (a) Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

   (b) Find the points (x, y) where the tangent plane to the graph of f is parallel to the $x - y$ plane. Be sure to explain your answer.

14. Consider the two vectors

$$\vec{v} = (3, -2, 1) \qquad \vec{u} = (2, 1, 4).$$

   (a) Compute $\vec{u} \times \vec{v}$.

   (b) Find a vector $\vec{w}$ which is perpendicular to $\vec{v}$, and explain why the answer you give is correct.

15. Consider the planes Π_1 and Π_2 , where Π_1 contains the point $p_1 = (3, 2, 1)$ and has the normal vector $\vec{n_1} = (2, 0, 1)$, while Π_2 is given by the equation $x + y - z = 3$.

   (a) Find the cosine of the angle θ between Π_1 and Π_2.

   (b) Write down the equation for Π_1.

   (c) Find a vector $\vec{v}$ which is parallel to both Π_1 and Π_2.

   (d) Parameterize the line l of intersection between Π_1 and Π_2.

16. Consider the parameterized curve

$$\vec{r}(t) = (e^t, t, e^{-t}).$$

   (a) points) Find the velocity vector of $\vec{r}$.

   (b) Is the tangent line to $\vec{r}$ ever parallel to the yz–plane? Be sure to explain your answer.

(c) Set up, but do not evaluate, the integral to compute to arclength of the section of $\vec{r}$ for $0 \leq t \leq 1$.

17. Consider the function $f(x, y) = e^{x^2+y^2-1}$.

    (a) Sketch the $\{f = 1\}$ level set.

    (b) For which values of z does the level set $\{f = z\}$ not contain any points? Be sure to explain your answer.

18. Consider the function $f(x, y) = xe^y + \cos(xy)$.

    (a) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

    (b) Recall that $(1, 0, \partial f/\partial x)$ and $(0, 1, \partial f/\partial y)$ are two tangent directions for the tangent plane to the graph of f at each of its points. Is this tangent plane ever parallel to the plane $x = y$? Be sure to explain your answer.