Exam 1

Math 210, Fall 06

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Last First

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PS #

To receive full credit you must show all your work. If you run out of room for an answer, continue on the back of the page.

This exam has 5 questions, for a total of 50 points.

Problem #	Grade
1	
2	
3	
4	
EC	
Total	

1. (a) (10 points) Find the intersection point P of the line $\mathbf{r}(t) = \langle 1+2t, 1-t, t\rangle$ and the plane Π given by the equation $x + 2y - z = 1$.

At the point of intersection, we have:

$$(1+2t) + 2(1-t) - t = 1 \quad \Longrightarrow \quad t = 2.$$

Therefore the intersection point is $(5, -1, 2)$.

(b) Is the line $\mathbf{r}$ perpendicular to the plane Π? Why?

No. Because the normal vector to the plane $\langle 1, 2, -1\rangle$ is not parallel to the vector $\langle 2, -1, 1\rangle$ which is paralle to the line $\mathbf{r}$.

(c) Find the equation of the line passing through the intersection point P above that is perpendicular to the plane Π.

$$\mathbf{l}(t) = \langle 5, -1, 2\rangle + t\,\langle 1, 2, -1\rangle\,.$$

2. (a) (10 points) Find the equation of the tangent line to the parametric curve

$$\mathbf{r}(t) = \left\langle e^{t-2}, \cos(\pi t), t-1 \right\rangle$$

at the point $(1,1,1)$.

If $\mathbf{r}(t) = \langle 1,1,1 \rangle$ then $t = 2$.

$$\mathbf{r}'(t) = \left\langle e^{t-2}, -\pi \sin(\pi t), 1 \right\rangle$$
$$\mathbf{r}'(2) = \langle 1,0,1 \rangle$$

Therefore, the equation of the tangent line at the point $(1,1,1)$ is:

$$\langle 1,1,1 \rangle + t \langle 1,0,1 \rangle$$

(b) Find the equation of the tangent plane to the graph of the function

$$f(x,y) = (xy + x^2)^2$$

at the point $(1,0,1)$.

$$f_x = 2(xy + x^2)(y + 2x) \implies f_x(1,0) = 4.$$
$$f_y = 2(xy + x^2)(x) \implies f_y(1,0) = 2.$$

The equation of the tangent plane is:

$$z - 1 = 4(x-1) + 2(y-0).$$

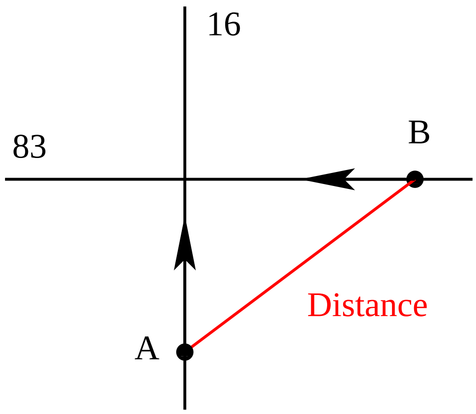
or

$$4x + 2y - z = 3.$$

3. (15 points) Car A is traveling north on Highway 16 and car B is traveling west on Highway 83. Each car is approaching the intersection of these Highways. Car A is 0.3 km from the intersection and is moving at $90\frac{km}{h}$ while car B is 0.4 km from the intersection and is moving $80\frac{km}{h}$. How fast is the distance between the cars changing at that moment?

Let

d be the distance between car A and car B
x be the distance from car B to the intersection
y be the distance from car A to the intersection



We have

$$d = \sqrt{x^2 + y^2}$$

The chain rule states:

$$\frac{\partial d}{\partial t} = \frac{\partial d}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial d}{\partial y}\frac{\partial y}{\partial t}$$

We also know:

$$x = 0.4, \qquad y = 0.3, \qquad \frac{\partial x}{\partial t} = -80, \qquad \frac{\partial x}{\partial t} = -90$$

$$\frac{\partial d}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{.4}{\sqrt{.16+.9}} = \frac{4}{5}$$

$$\frac{\partial d}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{.3}{\sqrt{.16+.9}} = \frac{3}{5}$$

Therefore

$$\frac{\partial d}{\partial t} = \frac{4}{5}(-80) + \frac{3}{5}(-90) = -118.$$

4. (a) (15 points) Find and classify all the critical points of the function

$$f(x, y) = x^3 + y^3 - 3xy.$$

First we find the critical points:

$$f_x = 3x^2 - 3y = 0 \implies x^2 = y \quad (1)$$
$$f_y = 3y^2 - 3x = 0 \implies y^2 = x \quad (2)$$

From equations (1) and (2) we get

$$x^4 = x \implies x(x^3 - 1) = 0 \implies x = 0 \quad \text{or} \quad x = 1$$

If $x = 0$ then $y = 0$ therefore $(0, 0)$ is a critical point.
If $x = 1$ then $y = 1$ therefore $(1, 1)$ is a critical point.

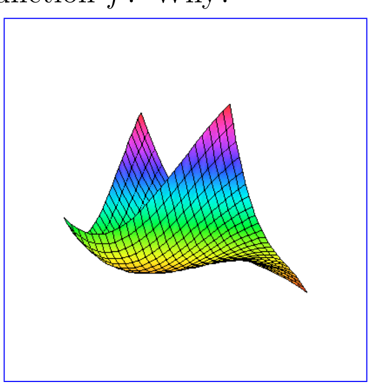
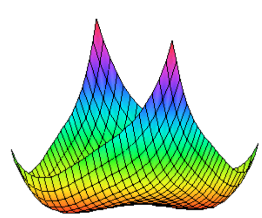
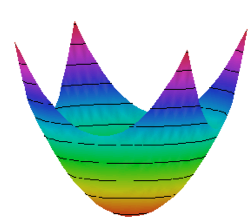
To classify the type of these critical points, note that:

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3 \quad \text{and} \quad \Delta = f_{xx} f_{yy} - (f_{xy})^2 = 36xy - 9.$$

At the critical point (1,1) $\Delta = 27 > 0$ $f_{xx} = 6 > 0$
(1,1) is a local mimimum.

At the critical point (0,0) $\Delta = -9 < 0$
(0,0) is a saddle point.

(b) Based on your answers above, which of the following could be the graph of the function f? Why?

It is the only one with one local minimum and one saddle point.

5. **Extra Credit:** Find the maximum and the minimum of

$$f(x, y) = x^2 - y^2,$$

over the region $D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$.

First we find the inside the domain D.

$$f_x = 2x = 0 \quad \Longrightarrow \quad x = 0.$$

$$f_y = -2y = 0 \quad \Longrightarrow \quad y = 0.$$

The point $(0, 0)$ is the only critical point.

Let $g(x, y) = x^2 + y^2$. The maximum on the boundary is at a point where $\nabla f = k\nabla g$.

$$\nabla f = \langle 2x, -2y \rangle, \qquad \nabla g = \langle 2x, 2y \rangle$$

$$\nabla f = k\nabla g \quad \Longrightarrow \quad \begin{cases} 2x = 2kx \\ 2y = -2ky \\ x^2 + y^2 = 1 \end{cases}$$

From $2x = 2kx$ we deduce that either $x = 0$ or $k = 1$.

If $x = 0$ then $0^2 + y^2 = 1 \quad \Longrightarrow \quad y = \pm 1$.

If $k = 1$ then $2y = -2y \quad \Longrightarrow \quad y = 0 \quad \Longrightarrow \quad x = \pm 1$.

Therefore, we have to check the point $(1-, 0)$, $(1, 0)$, $(0, -1)$ and $(0, 1)$ in the boundary

$$\begin{aligned} f(0, 0) &= 0 \\ f(-1, 0) &= 1 \\ f(1, 0) &= 1 \\ f(0, -1) &= -1 \\ f(0, 1) &= -1 \end{aligned}$$

The global maximum is 1 which occurs at points $(-1, 0)$ and $(1, 0)$.

The global maximum is -1 which occurs at points $(0, -1)$ and $(0, 1)$.