A computer-assisted uniqueness proof for a semilinear elliptic boundary value problem

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Uniqueness of solutions of the boundary value problem

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ p > 1, \ \lambda \in \mathbb{R} \end{cases} \tag{1}$$

Conjecture: If Ω is bounded and convex, then uniqueness holds as long as a solution of (1) exists.

Some early results

- if $N \geq 3$ solutions of (1) do not exist if Ω is starshaped and $f(u) = u^p + \lambda u,\ p \geq \frac{N+2}{N-2},\ \lambda \leq 0$

- When Ω is a ball the conjecture has been proved for the full range of the values of λ and p for which existence holds, mainly exploiting O.D.E. techniques. ($\lambda = 0$, relatively easy,$\lambda \neq 0$ the uniqueness in the ball is much more difficult to obtain and the complete result is spread in several papers (NiNuss85, Zhang9,Sri93, AdiYad9, AftPac03)

- Ω is not a ball very few results are available, and then only for the case $\lambda = 0$, i.e. $f(u) = u^p$ (Zou94 where domains close to a ball are considered, or that of Grossi, 00 where the exponent p is close to the critical Sobolev exponent and the region was convex and symmetric in N orthogonal directions, $N \geq 3$ in R^N)

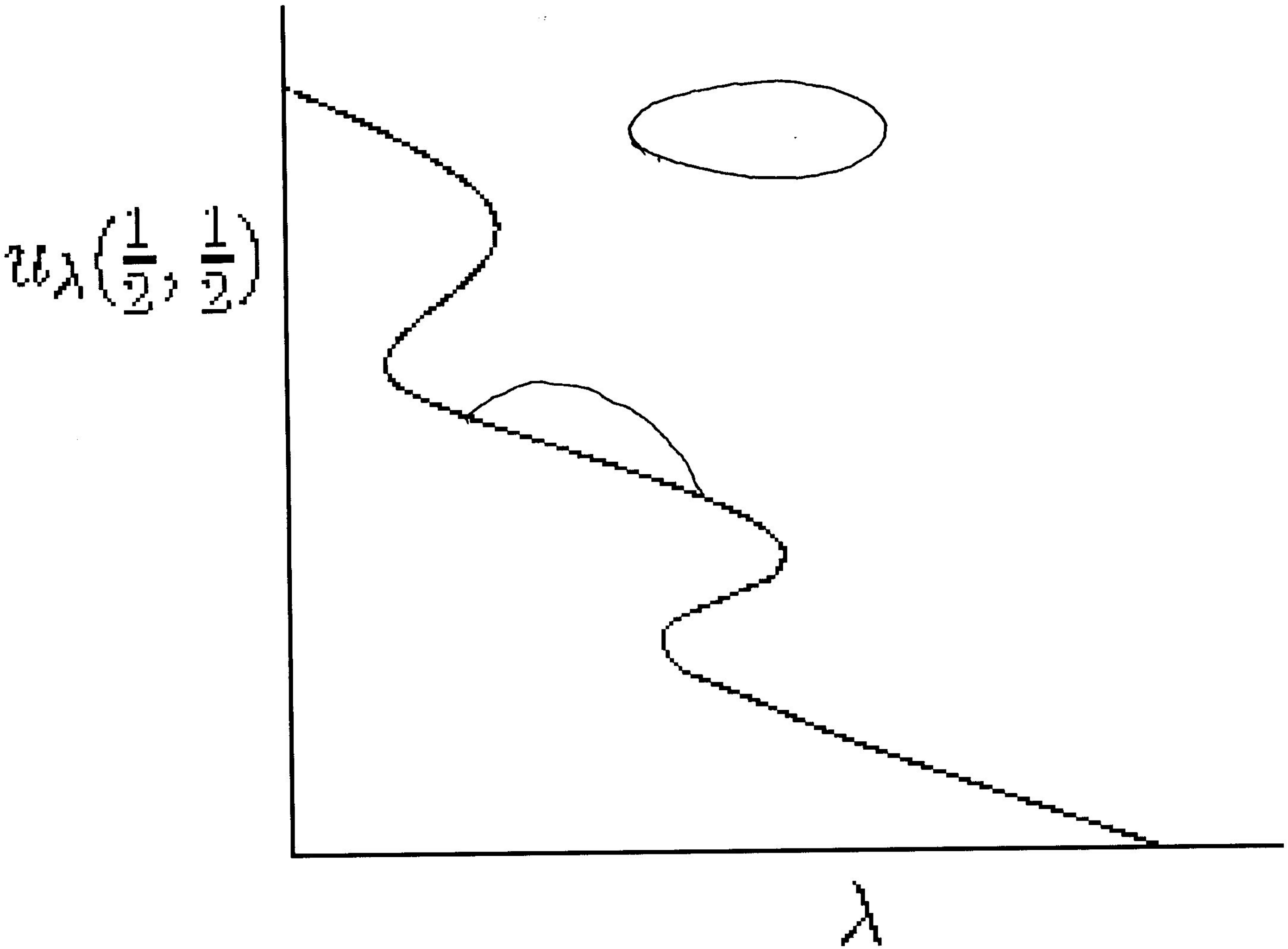
More general domains (non-ODE , $\lambda = 0$)

- (Lin 94) uniqueness is proved only for the so called "least-energy" positive solution, and the result holds in any bounded convex set in the plane

- (Dancer, 88) uniqueness for Ω a domain in $\mathbb{R}^2$, symmetric and convex in two orthogonal directions,(a continuation method and on the already known uniqueness result for the ball).

- (Damascelli, Grossi,Pacella 99) same result, but more information. A pure P.D.E. approach, based on the maximum principle, which does not rely on the uniqueness of the positive solution in the ball but indeed provides an independent proof for the ball.

The worst possible case



More general domains (non-ODE , $\lambda \neq 0$)

- Very little!

- Some qualitative information from Damascelli, Grossi,Pacella 99 on solutions of (1) for $u^p + \lambda u, \ \lambda < \lambda_1(\Omega)$ but not sufficient for uniqueness, due to difficulty in proving the nondegeneracy of solutions .

- (A solution u of (1) is said to be nondegenerate if the linearized operator $L_u = -\Delta - f'(u)$ does not admit zero as an eigenvalue in Ω with zero Dirichlet boundary conditions.)

Theorem 1. *Let* Ω *be the unit square in* $\mathbb{R}^2$, $\Omega = (0,1)^2$. *Then the problem*

$$\begin{cases} -\Delta u = u^p + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

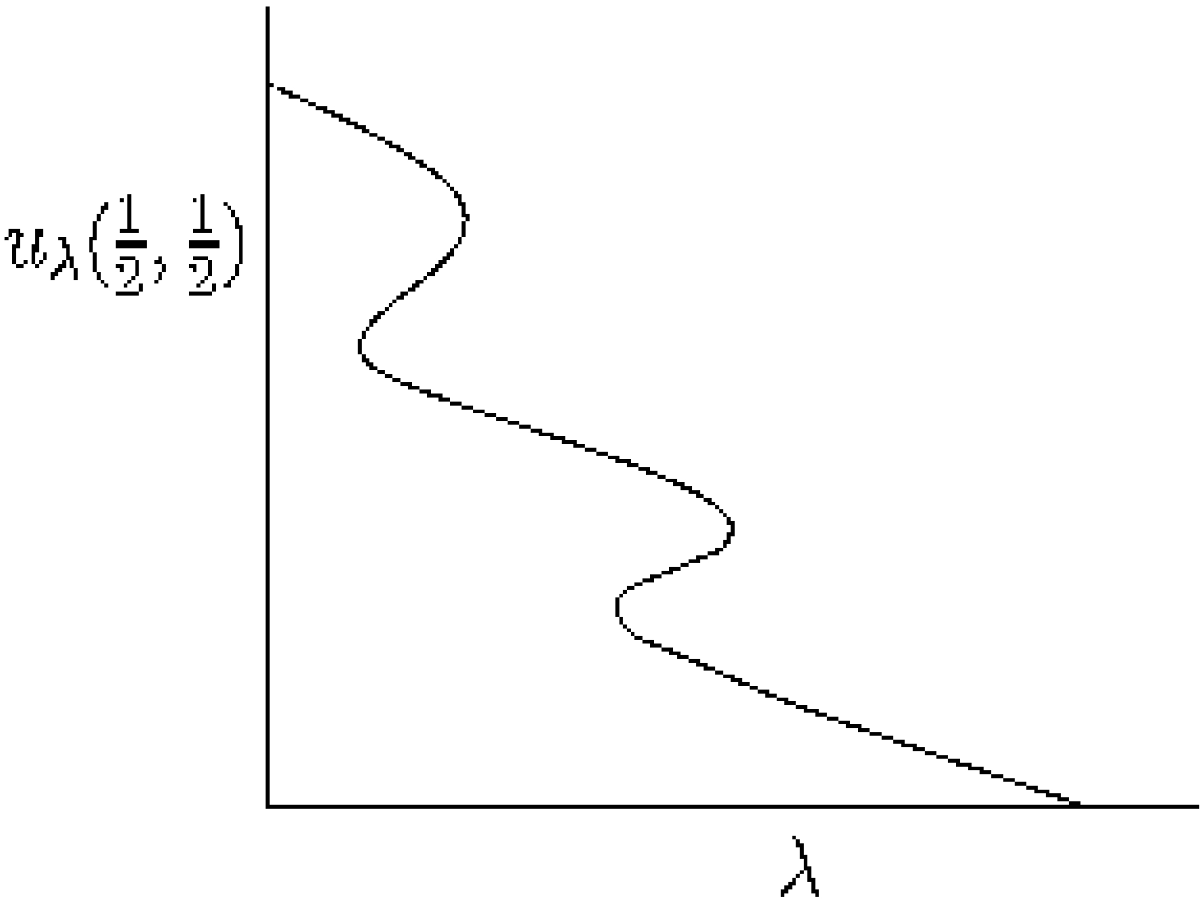
admits only one solution for any $\lambda \in [0, \lambda_1(\Omega))$ *and for* $p = 2$ *or* 3.

Two key ingredients.

- From Pacella-Shrikanth 03, and Damascelli, Grossi,Pacella 99, all solutions of (2) lie on a simple continuous curve, in the space $\mathbb{R} \times H_0^1(\Omega)$ joining the point $(\lambda_1, 0)$ with the point $(0, u_0)$, where u_0 is the unique positive solution of (**??**) for $\lambda = 0$.

- We are able to construct a branch of solutions connecting these two points and can show that along the branch solutions are nondegenerate, then uniqueness follows

Improvement as a result of last slide



The main analytic results

Proposition 1. *Assume that for some $\lambda \in (0, \lambda_1)$ and for all solutions u_λ of (2),*

$$\|u_\lambda\|_\infty \leq \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \equiv A. \tag{3}$$

Then, for that value of λ, problem (2) has only one solution which is also nondegenerate.

Proposition 2. *If there exists* $\bar{\lambda} \in (0, \lambda_1)$ *and a solution* $u_{\bar{\lambda}}$ *of (2) with* $\lambda = \bar{\lambda}$ *such that*

$$\|u_{\bar{\lambda}}\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{\lambda_1}\right)^{\frac{1}{p-1}} \equiv B \tag{4}$$

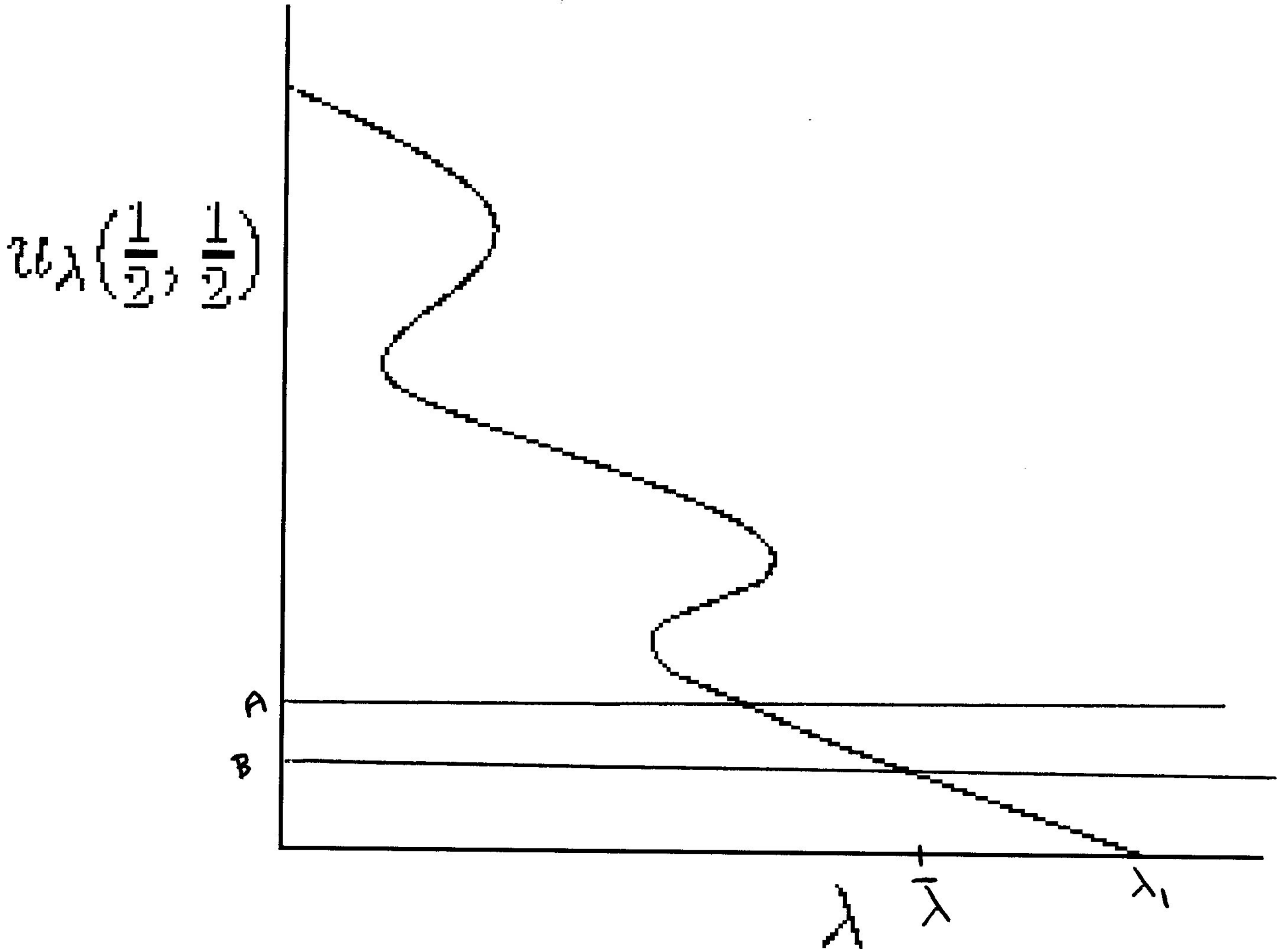
then

$$\|u_\lambda\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} = A \tag{5}$$

for all solutions u_λ *of (2) belonging to the branch* $\Gamma_2 \subset \Gamma$ *which connects* $(\bar{\lambda}, u_{\bar{\lambda}})$ *to* $(\lambda_1, 0)$*.*

(Γ being the unique continuous branch of solutions joining the point $(\lambda_1, 0)$ with the point $(0, u_0)$.)

The result of the last two propositions.



Corollary 1. *If on the branch* Γ *there exists a solution* $u_{\bar{\lambda}}, \bar{\lambda} \in (0, \lambda_1)$ *such that:*

i) on the sub-branch Γ_1 *connecting* $(0, u_0)$ *with* $(\bar{\lambda}, u_{\bar{\lambda}})$ *all solutions are nondegenerate*

and

$$ii) \qquad \|u_{\bar{\lambda}}\|_\infty < \left(\frac{\lambda_2 - \lambda_1}{p}\right)^{\frac{1}{p-1}} \cdot \left(\frac{\bar{\lambda}}{\lambda_1}\right)^{\frac{1}{p-1}}. \tag{6}$$

Then all solutions of equation (1) are nondegenerate, for all $\lambda \in (0, \lambda_1)$*, and therefore problem (1) admits only one solution for every* $\lambda \in [0, \lambda_1(\Omega))$*.*

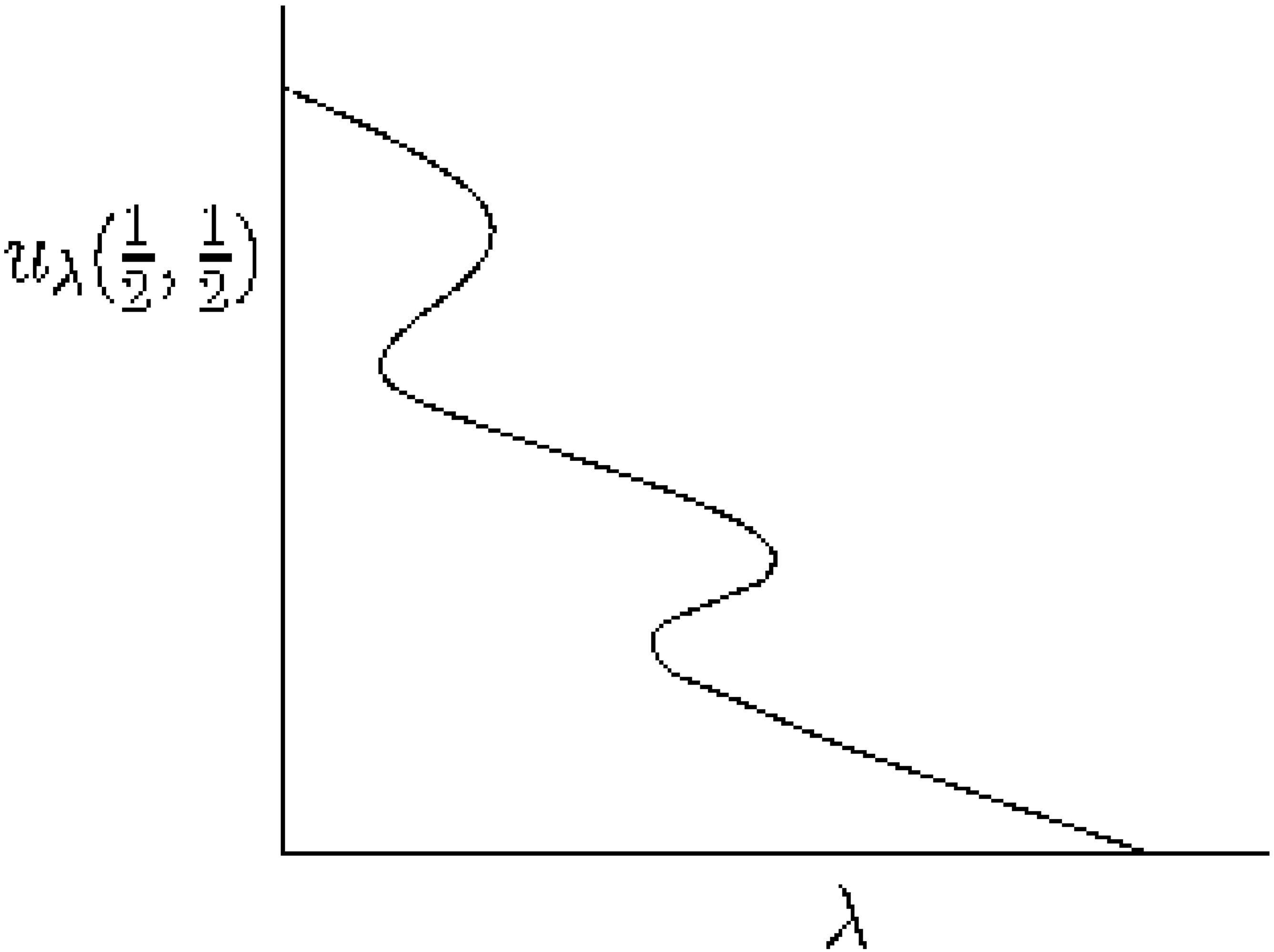
Now for some computer assistance!

Look for positive solutions of

$$-\Delta u = \lambda u + u^2 \quad \text{in} \quad \Omega = (0,1)^2, \qquad u = 0 \quad \text{on} \quad \partial\Omega \tag{7}$$

in $H_0^1(\Omega)$, endowed with $\langle u, v \rangle_{H_0^1} := \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx$.

But there *might* be turning points:



i.e. non-uniqueness of solutions for certain λ-ranges.

Aim now: Exclude turning points, and also secondary bifurcation points, by computer-assisted means, which implies *uniqueness* for the (positive) solution.

General outline

1) Compute continuous branch $\left\{ \begin{array}{ccc} [0, \lambda_1 - \eta] & \to & H_0^1(\Omega) \\ \lambda & \mapsto & \omega_\lambda \end{array} \right\}$ of *approximate* solutions

2) Compute defect bounds $\|\Delta\omega_\lambda + \lambda\omega_\lambda + \omega_\lambda^2\|_{H^{-1}} \leq \delta_\lambda$,
δ_λ piecewise constant and lower semi-continuous in λ

3) Compute bounds K_λ, piecewise constant and lower semi-continuous in λ, such that

$$\|v\|_{H_0^1} \leq K_\lambda \left\|L_{(\lambda,\omega_\lambda)}[v]\right\|_{H^{-1}} \quad \text{for all} \quad v \in H_0^1(\Omega),$$

where $L_{(\lambda,\omega_\lambda)}$ is the linearization of problem (1) at ω_λ. More general, for any $u \in H_0^1(\Omega)$, let

$$L_{(\lambda,u)} : \left\{ \begin{array}{ccc} H_0^1(\Omega) & \to & H^{-1}(\Omega) \\ v & \mapsto & -\Delta v - \lambda v - 2uv \end{array} \right\}$$

denote the linearization of (1) at u.

δ and K should be small!

Compute continuous branch $\left\{ \begin{array}{ccc} [0, \lambda_1 - \eta] & \to & H_0^1(\Omega) \\ \lambda & \mapsto & \omega_\lambda \end{array} \right\}$ of *approximate* solutions

i) Choose a *grid* $0 = \lambda^0 < \lambda^1 < \cdots < \lambda^N = \lambda_1 - \eta$.

ii) For each $i \in \{0, \ldots, N\}$, compute an approximation ω^i of problem (1) (with $\lambda = \lambda^i$) of the form

$$\omega^i(x) = \sum_{j,k=1}^{M} a_{jk}^i \sin(j\pi x_1) \sin(k\pi x_2) \tag{8}$$

by Newton's iteration and a Ritz-Galerkin method (for the linear sub-problems)

iii) Define $(\omega_\lambda)_{\lambda \in [0, \lambda_1 - \eta]}$ by piecewise linear interpolation of $\omega^0, \ldots, \omega^N$:
For $\lambda \in [\lambda^{i-1}, \lambda^i]$: $\omega_\lambda := \frac{\lambda^i - \lambda}{\lambda^i - \lambda^{i-1}} \, \omega^{i-1} + \frac{\lambda - \lambda^{i-1}}{\lambda^i - \lambda^{i-1}} \, \omega^i$

4) Check if $\boxed{\delta_\lambda < \frac{1}{4\gamma K_\lambda^2}}$ (where $\gamma := C_{p_1} C_{p_2} C_{p_3}, \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$,

$\|u\|_{L^p} \leq C_p \|u\|_{H_0^1}$ $(u \in H_0^1(\Omega))$).

Satisfied if δ_λ sufficiently small, i.e. if the approximations are sufficiently accurate!

In the affirmative case, we can prove the existence of a solution $u_\lambda \in H_0^1(\Omega)$ of problem (1) such that

$$\|u_\lambda - \omega_\lambda\|_{H_0^1} \leq \alpha_\lambda := \frac{2K_\lambda \delta_\lambda}{1 + \sqrt{1 - 4\gamma \delta_\lambda K_\lambda^2}} \tag{9}$$

(α_λ lower semi – continuous in λ).

Furthermore, we can show that

$$\left.\begin{array}{l} u \in H_0^1(\Omega) \quad \text{arbitrary} \\ \|u - \omega_\lambda\|_{H_0^1} \leq \alpha_\lambda \end{array}\right\} \Rightarrow L_{(\lambda,u)} : H_0^1(\Omega) \to H^{-1}(\Omega) \quad \text{bijective} \tag{10}$$

and, for some $\sigma > 0$,

$$\left.\begin{array}{l} u \in H_0^1(\Omega) \quad \text{solution of (1)} \\ \|u - \omega_\lambda\|_{H_0^1} \leq \max\{\alpha_\lambda, \alpha_{\lambda-0}\} + \sigma \end{array}\right\} \Rightarrow u = u_\lambda \tag{11}$$

Theorem 2. $(u_\lambda)_{\lambda \in [0, \lambda_1 - \eta]}$ *is a smooth solution branch for problem (1), and*

$$L_{(\lambda, u_\lambda)} : H_0^1(\Omega) \to H^{-1}(\Omega) \text{ is bijective for each } \lambda \in [0, \lambda_1 - \eta]. \quad (12)$$

Corollary 2. $(u_\lambda)_{\lambda \in [0, \lambda_1 - \eta]}$ *is* **the** *solution (sub-)branch known analytically already, and by (5),* u_λ *is non-degenerate for* $\lambda \in [0, \lambda_1 - \eta]$*.*

By the Implicit Function Theorem, there are no turning points or secondary bifurcation points on this sub-branch.

How to treat the remaining interval $[\lambda_1 - \eta, \lambda_1]$?

If, for some $\bar{\lambda} \in (0, \lambda_1)$,

$$u_{\bar{\lambda}}\left(\frac{1}{2}, \frac{1}{2}\right) < \frac{3}{4}\bar{\lambda} \qquad (= B), \tag{13}$$

then the (positive) solution is unique and non-degenerate for all $\lambda \in [\bar{\lambda}, \lambda_1)$.
Thus we have to choose $\eta > 0$ such that (12) holds for $\bar{\lambda} = \lambda_1 - \eta$.

Theorem 3. *The positive solution of (1) is unique for all $\lambda \in [0, \lambda_1)$.*

Suppose that $\delta_\lambda < \frac{1}{4\gamma K_\lambda^2}$ for all $\lambda \in [0, \lambda_1 - \eta\ ]$ (*finitely* many inequalities to check)

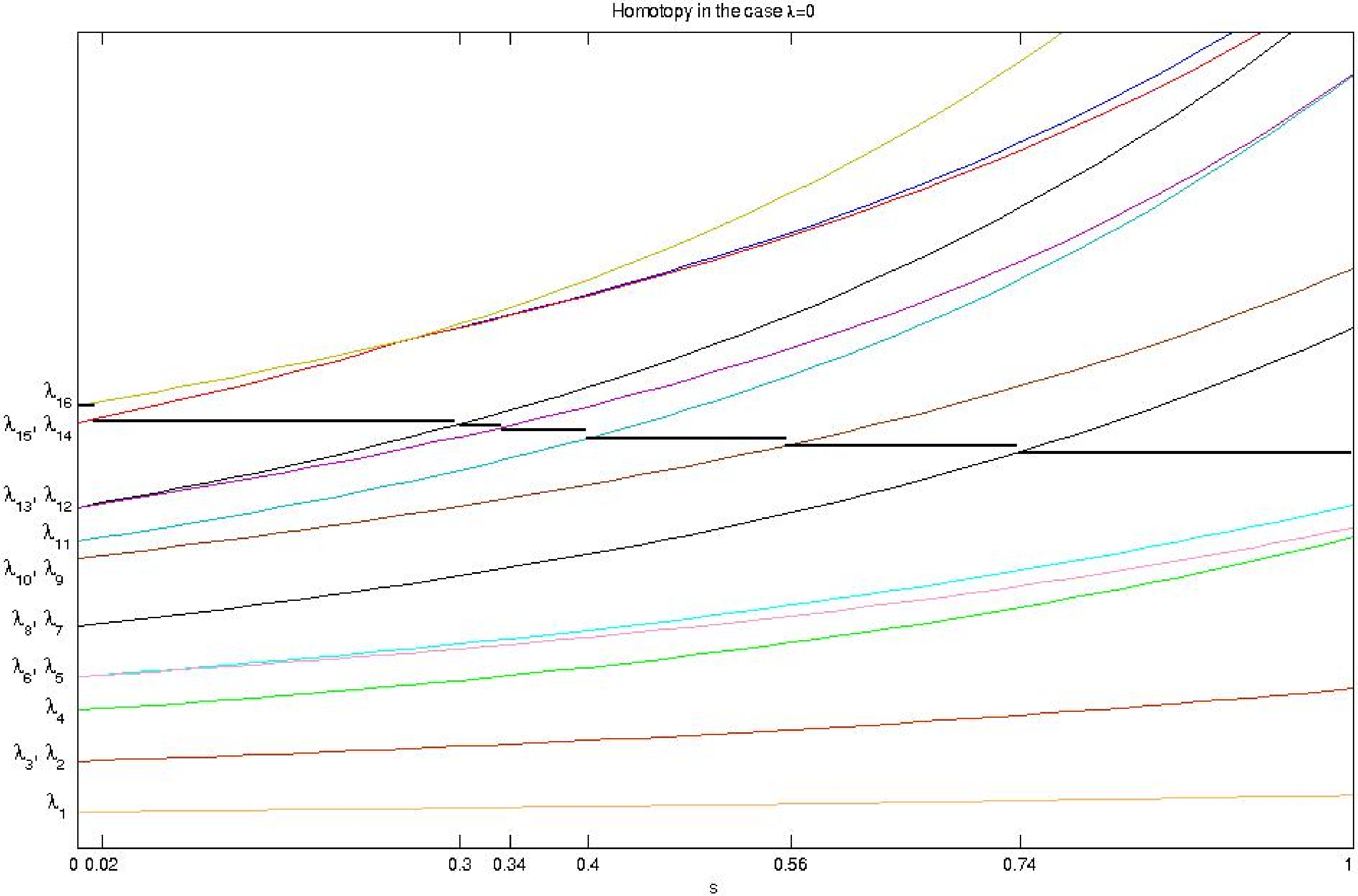
Theorem 4. *For each $\lambda \in [0, \lambda_1 - \eta]$, there exists a solution $u_\lambda \in H_0^1(\Omega)$ of problem (1) such that*

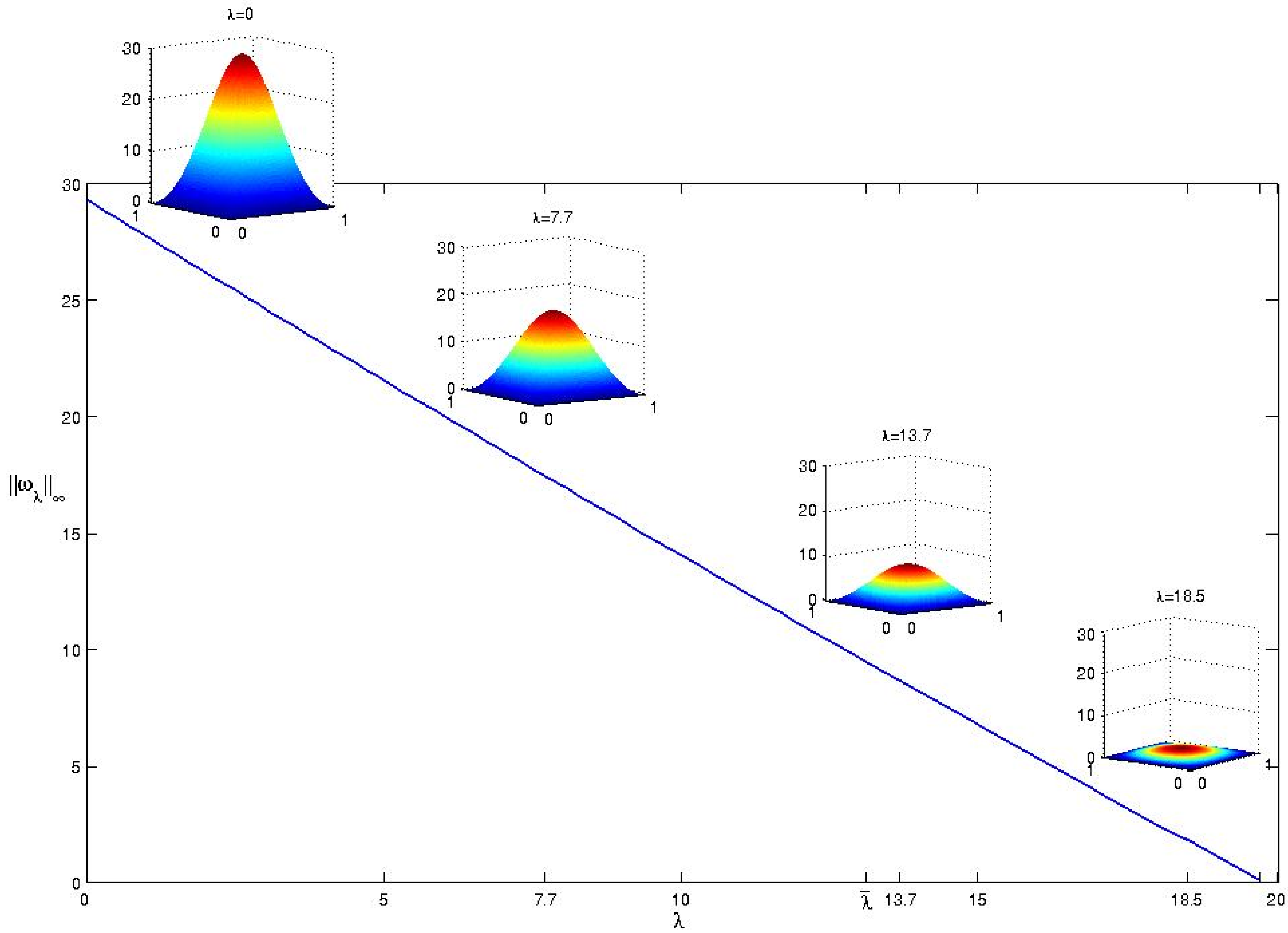
$$\|u_\lambda - \omega_\lambda\|_{H_0^1} \leq \alpha_\lambda := \frac{2K_\lambda \delta_\lambda}{1 + \sqrt{1 - 4\gamma\delta_\lambda K_\lambda^2}}, \qquad (2)$$

$$\left.\begin{array}{l} u \in H_0^1(\Omega) \quad \text{arbitrary} \\ \|u - \omega_\lambda\|_{H_0^1} \leq \alpha_\lambda \end{array}\right\} \Rightarrow L_{(\lambda,u)} : H_0^1(\Omega) \to H^{-1}(\Omega) \quad \text{bijective} \qquad (3)$$

and, for some $\sigma > 0$,

$$\left.\begin{array}{l} u \in H_0^1(\Omega) \quad \text{solution of (1)} \\ \|u - \omega_\lambda\|_{H_0^1} \leq \max\{\alpha_\lambda, \alpha_{\lambda-0}\} + \sigma \end{array}\right\} \Rightarrow u = u_\lambda \qquad (4)$$


Homotopy in the case λ=0
λ16
λ15, λ14
λ13, λ12
λ11
λ10, λ9
λ8, λ7
λ6, λ5
λ4
λ3, λ2
λ1
0 0.02
0.3
0.34
0.4
0.56
0.74
1
s



λ=0
λ=7.7
λ=13.7
λ=18.5
$||\omega_\lambda||_\infty$
λ
0
5
7.7
10
$\bar{\lambda}$
13.7
15
18.5
20

grid: $0 = \lambda^0 < 0.1 < 0.3 < 0.5 < \ldots < 18.3 < 18.5 = \lambda^{94} = 2\pi^2 - \eta$ $(N = 94)$

for $i \in \{0, \ldots, 94\}$: $\omega_i = \sum_{j,k}^{15} a^i_{jk} \sin(j\pi x_1) \sin(k\pi x_2)$ $\qquad (M = 15)$

λ interval	α_λ	K_λ	δ_λ
$[0, 05]$	0.15161	2.89603	0.05158
$(2, 2.2)$	0.14424	2.77827	0.05123
$(4, 4.2)$	0.11589	2.61615	0.04385
$(6, 6.2)$	0.09138	2.46905	0.03674
$(8, 8.2)$	0.07048	2.33536	0.03002
$(10, 10.2)$	0.05343	2.23154	0.02385
$(12, 12.2)$	0.05270	2.85105	0.01839
$(14, 14.2)$	0.05498	3.94930	0.01382
$(16, 16.2)$	0.06722	6.42361	0.01032
$(18, 18.2)$	0.15124	17.1357	0.00807
$(18.4, 18.5]$	0.25574	25.6786	0.00779

$$\tau_i = \frac{1}{4\sqrt{2\pi^2+1}}(\lambda^i - \lambda^{i-1})||\omega^i - \omega^{i-1}||_\infty, \quad \rho_i = \frac{1}{4\sqrt{2\pi^2+1}}||\omega^i - \omega^{i-1}||_\infty^2$$

i	τ_i	ρ_i
1	0.00087	0.00138
10	0.00344	0.00539
20	0.00339	0.00523
30	0.00334	0.00508
40	0.00329	0.00494
50	0.00324	0.00480
60	0.00320	0.00468
70	0.00316	0.00455
80	0.00312	0.00444
90	0.00308	0.00433
94	0.00307	0.00429