Name:__________________

Math 2410
Final Exam May 6 2008

You must show your work to receive credit.

1. Consider the autonomous differential equation

$$y' = -y(y-3)^2(y-5).$$

   (a) Compute the equilibrium solutions.

   (b) Sketch the phase line and classify the equilibria as sinks, sources, or nodes.

   (c) Describe the long term behavior of the solution to the above differential equation with initial condition $y(0) = 2$.

2. Consider the initial value problem:

$$\begin{aligned} y' &= t^{\frac{1}{3}}(y-1)^{\frac{4}{3}} \\ y(t_0) &= y_0 \end{aligned}$$

Do not solve the differential equation.

(a) If $t_0 = 5$ and $y_0 = 0$, is the existence of a unique solution guaranteed? *Why or why not?*

(b) If $t_0 = 0$ and $y_0 = 1$, is the existence of a unique solution guaranteed? *Why or why not?*

3. Find the general solution of

$$y' + 2y = \sin(t)$$

Solve the initial value problem

$$\begin{aligned} y' - (3/t)y &= t^2 \\ y(1) &= 3. \end{aligned}$$

4. A container holds 50 gals. of pure water. A salt solution containing 1 lb. per gal. enters at the rate of 2 gals a minute and the well-stirred mixture leaves at the same rate. Find the amount of salt in the tank at time t.

5. Consider the linear system $\vec{Y}' = A\vec{Y}$ where $\vec{Y} = (x, y)$ and

$$A = \begin{pmatrix} 5 & 4 \\ 9 & 0 \end{pmatrix}$$

(a) Classify the equilibrium at the origin (sink, spiral source, etc). Explain your answer.

(b) What is the general solution to the system? Sketch the phase plane.

(c) Solve the initial value problem with initial condition $\vec{Y}(0) = (2, 3)$.

6. Given the matrix

$$B = \begin{pmatrix} -2 & 5 \\ -5 & -2 \end{pmatrix}$$

Find one (complex) eigenvalue and a corresponding eigenvector.

Let A be a real 2×2 matrix with complex eigenvalues $1 + 2i$ and $1 - 2i$. Assume that $\vec{V} = (1 + i, 1)$ is an eigenvector corresponding to eigenvalue $1 + 2i$.

Find the general solution to the system of differential equations $\frac{dY}{dt} = AY$

Find the solution to the system with initial condition $\vec{Y}(0) = (2, 3)$.

Is the origin a spiral sink, a spiral source or a center? Why?

7. Consider the spring-mass system whose motion is governed by

$$y'' + 6y' + 34y = 34t$$

(a) Compute the general solution to the above equation.

(b) Describe the long term behavior of the mass.

8. Find the general solution for the spring-mass problem $y'' + 9y = \sin(3t)$. Solve with initial conditions $y(0) = 0, y'(0) = 1$.

9. Consider the equation $y' + 4y = t - u_3(t)(t-3)$ with initial conditions $y(0) = 1$. Using the Laplace transform, find $y(t)$.

10. Consider the spring-mass system whose motion is governed by $y'' + 4y = f(t)$ with initial conditions $y(0) = 0, y'(0) = 0$ where the function $f(t)$ is given by $1 - u_2(t)(t - 2)$.

Using the Laplace transform, find $y(t)$.