Mathematics 114Q
Practice Exam 2, Spring 2005

Name: ______________________________

Answer the questions in the space provided. If you need more space, continue on the back of the page. All answers must include supporting work or an explanation of your reasoning. Without supporting work, even correct answers may be marked wrong. Use of a calculator is not allowed. This practice exam has 12 questions, for a total of 150 points.

1. (45 points) Calculate the following integrals.

(a) $\int x \cos(x^2)\, dx$ $\qquad [u = x^2, du = 2\, dx]$

$$= \frac{1}{2}\int \cos(u)\, du = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin(x^2) + C.$$

(b) $\int x^2 \cos(x)\, dx$ $\qquad [u = x^2, dv = \cos(x)\, dx]$

$$\begin{aligned} &= x^2 \sin(x) - \int 2x \sin(x)\, dx \\ &= x^2 \sin(x) - 2\Big[- x\cos(x) - \int - \cos(x)\, dx\Big] \\ &= x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C \end{aligned}$$

(c) $\int \cos^2(x)\, dx$ $\qquad [u = \cos(x), dv = \cos(x)\, dx]$

$$\begin{aligned} &= \cos(x)\sin(x) + \int \sin^2(x)\, dx \\ &= \cos(x)\sin(x) + \int \Big(1 - \cos^2(x)\Big)\, dx \\ &= \cos(x)\sin(x) + x - \int \cos^2(x)\, dx. \end{aligned}$$

So

$$\int \cos^2(x)\, dx = \frac{1}{2}\Big(\cos(x)\sin(x) + x\Big) + C.$$

(d) $\displaystyle\int \frac{x+2}{x^2+x}\,dx$

$$= \int \left(\frac{2}{x} + \frac{-1}{x+1}\right)\,dx = 2\ln|x| - \ln|x+1| + C.$$

(e) $\displaystyle\int \frac{1}{x^3-4x}\,dx$

$$= \int \frac{1}{x(x+2)(x-2)}\,dx = \int \left(\frac{-1/4}{x} + \frac{1/8}{x+2} + \frac{1/8}{x-2}\right)\,dx$$

$$= -\frac{1}{4}\ln|x| + \frac{1}{8}\ln|x+2| + \frac{1}{8}\ln|x-2| + C.$$

(f) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$

$$= \arcsin\left(\frac{x}{3}\right) + C.$$

Use $x = 3\sin\theta$.

(g) $\displaystyle\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx$

$$= \int \left(\frac{A}{x-2} + \frac{Bx+C}{x^2+1}\right)\,dx = \int \left(\frac{1}{x-2} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\right)\,dx$$

$$= \ln|x-2| + \frac{1}{2}\ln(x^2+1) + \arctan(x) + C.$$

(h) $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$

$$= \int \frac{1}{(x+1)^2+1}\,dx = \arctan(x+1) + C.$$

[Use $x+1 = \tan\theta$]

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(i) $\displaystyle\int \frac{3}{\sqrt{2x-x^2}}\,dx$

$$= \int \frac{3}{\sqrt{1-(x-1)^2}}\,dx = 3\arcsin(x-1) + C.$$

[Use $x-1 = \sin\theta$]

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2. (30 points) Determine whether or not the following improper integrals converge. Evaluate, if possible.

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$$= \lim_{r\to 0}\int_r^1 x^{-1/2}\,dx = \lim_{r\to 0}\left.\frac{x^{1/2}}{1/2}\right|_r^1 = \lim_{r\to 0}\left.2\sqrt{x}\right|_r^1$$

$$= \lim_{r\to 0} 2\left(1-\sqrt{r}\right) = 2.$$

(b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x^4+x}}\,dx$

For $x > 0$, we have $x^4 + x > x$. It follows that

$$\frac{1}{\sqrt{x^4+x}} < \frac{1}{\sqrt{x}}.$$

Since $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is known to converge, it follows that $\displaystyle\int_0^1 \frac{1}{\sqrt{x^4+x}}\,dx$ converges, by the comparison test.

(c) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^4+x}}\,dx$

For $x > 0$, we have $x^4 + x > x^4$. It follows that

$$\frac{1}{\sqrt{x^4+x}} < \frac{1}{\sqrt{x^4}} = \frac{1}{x^2}.$$

Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is known to converge, it follows that $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^4+x}}\,dx$ converges, by the comparison test.

(d) $\int_e^\infty \frac{1}{(\ln x)^2} \cdot \frac{1}{x}\, dx$

Use the substitution: $u = \ln x$, $du = \frac{1}{x}\, dx$. Using this substitution, the limits change: when $x = e$, $u = 1$ and when $x \to \infty$, $u \to \infty$. So

$$\int_e^\infty \frac{1}{(\ln x)^2} \cdot \frac{1}{x}\, dx \;=\; \int_1^\infty \frac{1}{u^2}\, du \;=\; \lim_{R\to\infty} \int_1^R \frac{1}{u^2}$$

$$= \lim_{R\to\infty} -\frac{1}{u}\Big|_1^R \;=\; \lim_{R\to\infty} -\frac{1}{R} + 1 \;=\; 1.$$

(e) $\int_0^\infty e^{-x}\, dx$

$$\lim_{R\to\infty} \int_0^R e^{-x}\, dx = \lim_{R\to\infty} -e^{-x}\Big|_0^R = \lim_{R\to\infty} -e^{-R} + 1 = 1$$

(f) $\int_0^\infty e^{-x^2}\, dx = \int_0^1 e^{-x^2}\, dx + \int_1^\infty e^{-x^2}\, dx$

Part 1: $\int_0^1 e^{-x^2}\, dx < \infty$ by the Fundamental Theorem of Calculus since the integrand e^{-x^2} is continuous on [0,1].

Part 2: For $x > 1$, we have $x^2 > x$, so $e^{-x^2} < e^{-x}$. Since $\int_1^\infty e^{-x}\, dx$ is known to converge, we have $\int_0^1 e^{-x^2}\, dx$ converges, by the comparison test.

3. (15 points) Find the exact area under the following curves:

(a) $y = \frac{1}{x^{3/2}}$ for $x \geq 1$.

$$\texttt{Area} = \int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^{3/2}}\,dx = \lim_{R\to\infty} \frac{x^{-1/2}}{-1/2}\Big|_1^R$$

$$= \lim_{R\to\infty} -\frac{2}{\sqrt{x}}\Big|_1^R = \lim_{R\to\infty} -\frac{2}{\sqrt{R}} + 2 = 2.$$

(b) $y = xe^{-2x}$ for all $x \geq 0$. [Use integration by parts with $u = x$ and $dv = e^{-2x}\,dx$.]

$$\texttt{Area} = \int_0^\infty xe^{-2x}\,dx = \lim_{R\to\infty} \int_0^R xe^{-2x}\,dx$$

$$= \lim_{R\to\infty} \Big[-\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx \Big]\Big|_0^R = \lim_{R\to\infty} \Big[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\Big]\Big|_0^R = \frac{1}{4}.$$

(c) The graph of y, where $y = \frac{1}{\sqrt{x}}$ for $x \leq 1$ and $y = \frac{1}{x^2}$ for $x > 1$.

$$\texttt{Area} = \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^\infty \frac{1}{x^2}\,dx = 2 + 1 = 3.$$

4. (15 points) Find the area of the following regions:

(a) The region bounded by $y = x$ and $y = \sqrt{x}$.

$$\int_0^1 \left(x^{1/2} - x\right)\, dx = \frac{1}{6}.$$

(b) The region bounded by $y = \sqrt{x}$ and $y = x^2$.

$$\int_0^1 \left(x^{1/2} - x^2\right)\, dx = \frac{7}{6}.$$

(c) The region bounded by $x = 0$, $y = 1 - x$ and $y = x^2 - 1$.

$$\int_0^1 \left((1 - x) - (x^2 - 1)\right)\, dx = \int_0^1 \left(2 - x - x^2\right)\, dx = \frac{1}{6}.$$

5. (10 points) Find the following arc lengths:

(a) The curve $f(x) = \frac{2}{3}\sqrt{x^3}$ from $x = 0$ to $x = 1$.

`Observe,` $f'(x) = \dfrac{d}{dx}\left(\dfrac{2}{3}\,x^{3/2}\right) = x^{1/2}$. `So the arc length is:`

$$\int_0^1 \sqrt{1 + f'(x)^2}\,dx = \int_0^1 \sqrt{1 + \left(x^{1/2}\right)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx$$

$$= \frac{2}{3}(2^{3/2} - 1) = \frac{2}{3}(2\sqrt{2} - 1).$$

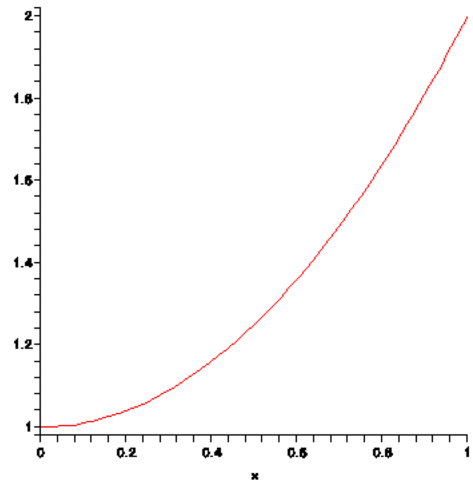
(b) The parametric curve $x = \sin(e^t)$, $y = \cos(e^t)$, from $t = 0$ to $t = 1$.

$$\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{\left(\cos(e^t)\cdot e^t\right)^2 + \left(-\sin(e^t)\cdot e^t\right)^2}\,dt$$

$$= \int_0^1 \sqrt{e^{2t}\left(\cos^2(e^t) + \sin^2(e^t)\right)}\,dt = \int_0^1 e^t\,dt = e - 1.$$

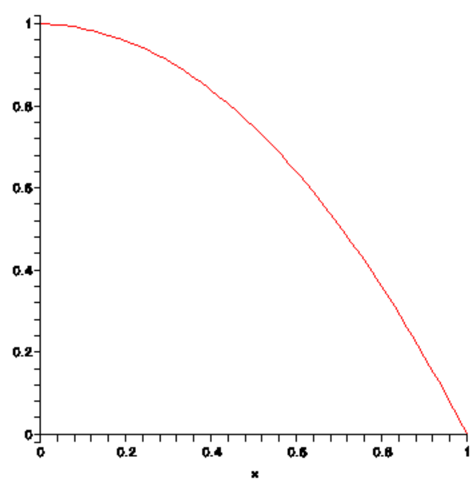
6. (5 points) Find the volume of the solid region obtained by rotating the region bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ around the x-axis.

$$\int_0^1 \pi\left(x^2+1\right)^2\,dx = \int_0^1 \pi\left(x^4+2x^2+1\right)\,dx = \frac{28}{15}\pi.$$



7. (5 points) Find the volume of the solid region obtained by rotating the region bounded by $y = 1 - x^2$, $y = 0$, $x = 0$, and $x = 1$ around the x-axis.

$$\int_0^1 \pi\left(1-x^2\right)^2\,dx = \int_0^1 \pi\left(1-2x^2+x^4\right)\,dx = \frac{8}{15}\pi.$$



8. (5 points) Find the mass of a rod of length 5 cm with density $\delta(x) = 1 + x^2$ gm/cm at a distance of x cm from the left end.

$$\int_0^5 (1 + x^2)\, dx = x + \frac{1}{3}x^3 \Big|_0^5 = 5 + \frac{5^3}{3} = \frac{140}{3}.$$

9. (5 points) Find the mass of a rod of length 10 cm with density $\delta(x) = \frac{1}{1+x}$ gm/cm at a distance of x cm from the left end.

$$\int_0^{10} \frac{1}{1+x}\, dx = \ln|1 + x| \Big|_0^{10} = \ln 11.$$

10. (5 points) A water tank has dimensions (in meters) 10 long, 15 high, and 20 wide. If the tank is full, how much work does it take to pump all the water out?

Recall that the weight of water is $9800\ kg/m^3$. Then the total work is given by the following integral:

$$\int_0^{15} 9800 \times 10 \times 20 \times (15 - x)\, dx = 1960000 \int_0^{15} (15 - x)\, dx$$

$$= 1960000 \times \frac{15^2}{2} \quad = \quad 220500000.$$

11. (5 points) A water tank has dimensions (in meters) 20 long, 10 high, and 15 wide. If the tank is full, how much work does it take to pump all the water out?

$$\int_0^{10} 9800 \times 15 \times 20 \times (10 - x)\, dx = 2940000 \int_0^{10} (10 - x)\, dx$$

$$= 2940000 \times \frac{10^2}{2} \quad = \quad 147000000.$$

12. (5 points) A water tank has dimensions (in meters) 10 long, 20 high, and 15 wide. If the tank is full, how much work does it take to pump all the water out?

$$\int_0^{20} 9800 \times 10 \times 15 \times (20 - x)\, dx = 1470000 \int_0^{20} (20 - x)\, dx$$

$$= 1470000 \times \frac{20^2}{2} \quad = \quad 294000000.$$