# RESEARCH STATEMENT

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## 1. Introduction

My research interests span algebra, number theory, and computational geometry and topology. In particular, my work has led to two separate Ph.D. dissertations, one in the mathematics department and one in the computer science department. This statement is organized into three sections:

- Witt-Burnside Rings,
- Manifold Approximation,
- Incidence Algebras.

## Witt-Burnside Rings

Classical Witt vectors [29] are a functorial construction which takes perfect fields $k$ of characteristic $p$ to domains $\mathbf{W}(k)$ of characteristic 0 with residue field $k$. This is surprising, as most functorial operations in algebra take rings of characteristic $p$ to other rings of characteristic $p$. The ring of classical Witt vectors $\mathbf{W}(k)$ is a minimal example of a complete discrete valuation ring unramified at $p$ with perfect residue field $k$. These are of interest in number theory [28, 29], Galois theory [15], algebraic groups [30], combinatorics [22], commutative algebra [14], and $K$-theory [13].

The construction of the classical Witt vectors uses the Witt polynomials

$$\sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^{n-1}X_{n-1}^{p} + p^n X_n$$

for $n \geq 0$ to define certain addition and multiplication polynomials with rational coefficients which non-obviously have integral coefficients. Dress and Siebeneicher generalized the Witt polynomials to a family of multivariable polynomials associated to any profinite group $G$ [8]. To define these generalized Witt polynomials, we use the isomorphism classes of finite transitive $G$-sets, called the *frame* of $G$ and denoted $\mathcal{F}(G)$. For example, $\mathcal{F}(\mathbf{Z}_p) = \mathbf{N}$ by the correspondence $p^n\mathbf{Z}_p \leftrightarrow n$. A natural partial ordering on $\mathcal{F}(G)$ is: for $T$ and $U$ in $\mathcal{F}(G)$ set $T \geq U$ when there is a $G$-map from $T$ to $U$.

Setting $\varphi_T(U)$ to be the number of $G$-maps from $T$ to $U$, the $T$-th *Witt polynomial* is

$$W_T(\{X_U\}_{U\in\mathcal{F}(G)}) = \sum_{U\leq T} \varphi_T(U) X_U^{\#T/\#U} = X_0^{\#T} + \ldots + \varphi_T(T)X_T,$$

where 0 denotes the trivial $G$-set $G/G$. If $G$ is abelian, the Witt polynomials simplify: $\varphi_T(U) = \#U$ when $U \leq T$, so

$$W_T(\{X_U\}_{U\in\mathcal{F}(G)}) = \sum_{U\leq T} (\#U) X_U^{\#T/\#U} = X_0^{\#T} + \ldots + (\#T)X_T.$$

Set $\underline{X} = \{X_T\}_{T\in\mathcal{F}(G)}$ and $\underline{Y} = \{Y_T\}_{T\in\mathcal{F}(G)}$. Since $W_T(\underline{X})$ is linear in the $X_T$-term we can recursively solve for the unique families of polynomials $\{S_T(\underline{X},\underline{Y})\}_{T\in\mathcal{F}(G)}$ and $\{M_T(\underline{X},\underline{Y})\}_{T\in\mathcal{F}(G)}$ in $\mathbf{Q}[\underline{X},\underline{Y}]$ satisfying

$$W_T(\underline{X}) + W_T(\underline{Y}) = W_T(\underline{S}) \text{ and } W_T(\underline{X})W_T(\underline{Y}) = W_T(\underline{M}) \text{ for all } T \in \mathcal{F}(G).$$

It is clear that $S_T$ and $M_T$ each only depend on the variables $X_U$ and $Y_U$ for $U \le T$ and have $\mathbf{Q}$-coefficients. A significant theorem of Dress and Siebeneicher [8, p. 107], which generalizes Witt's theorem (which is the case $G = \mathbf{Z}_p$ as an additive pro-$p$ group), says that the polynomials $S_T$ and $M_T$ have coefficients in $\mathbf{Z}$, and so can be used to define addition and multiplication on sequences $\mathbf{a} = (a_T)_{T\in\mathcal{F}(G)}$ indexed by $\mathcal{F}(G)$ with coordinates in any ring $A$, by setting

$$\mathbf{a}+\mathbf{b} = (S_T(\mathbf{a},\mathbf{b}))_{T\in\mathcal{F}(G)} \text{ and } \mathbf{a}\cdot\mathbf{b} = (M_T(\mathbf{a},\mathbf{b}))_{T\in\mathcal{F}(G)}.$$

These operations make the set of sequences $\mathbf{a} = (a_T)_{T\in\mathcal{F}(G)}$ into a ring denoted $\mathbf{W}_G(A)$. These are the generalized Witt vectors. The rings $\mathbf{W}_{\mathbf{Z}_p}(A), \mathbf{W}_{\mathbf{Z}/p^n\mathbf{Z}}(A)$ and $\mathbf{W}_{\widehat{\mathbf{Z}}}(A)$ are, respectively, the classical Witt vectors, the classical truncated Witt vectors, and Cartier's "big" Witt vectors. Category-theoretic properties of $\mathbf{W}_G$ have been explored by Elliott [9] and $\mathbf{W}_G(\mathbf{Z}[X_1, X_2, \ldots])$ has been embedded as subring of the unitary cobordism ring of $G$-manifolds [5]. Not much about the ring structure of $\mathbf{W}_G(k)$ for $G \neq \mathbf{Z}_p, \mathbf{Z}/p^n\mathbf{Z}$ or $\widehat{\mathbf{Z}}$ has been previously studied.

Motivated by the importance of classical Witt vectors in number theory, I have examined the following question:

**Question:** What can be said about the ring-theoretic properties of $\mathbf{W}_G(k)$ when $G$ is a pro-$p$ group and $k$ is a field of characteristic $p$?

For example, when $k$ is a perfect field of characteristic $p$, $\mathbf{W}_{\mathbf{Z}_p}(k) = \mathbf{W}(k)$ is the ring of classical Witt vectors, which is a $p$-adically complete discrete valuation ring with maximal ideal $(p)$ and residue field $k$. The ring $\mathbf{W}_G(k)$ for infinite pro-$p$ groups $G \ncong \mathbf{Z}_p$ is a local ring of characteristic 0 with maximal ideal $\mathfrak{m} = \{\mathbf{a} : a_0 = 0\}$ as in the classical case, but besides this the situation is much more complicated as indicated in the table below.

| Classical $\mathbf{W}_G(k)$, $G = \mathbf{Z}_p$ | General $\mathbf{W}_G(k)$, $G \ncong \mathbf{Z}_p$ |
|---|---|
| local, characteristic 0 | local, characteristic 0 when $G$ is infinite |
| discrete valuation ring | Not a domain, but is reduced for $G = \mathbf{Z}_p^d$, $d \ge 2$ |
| Maximal ideal $\mathfrak{m} = (p)$ | Maximal ideal $\mathfrak{m}$ not finitely generated for $G = \mathbf{Z}_p^d$, $d \ge 2$ |

Table 1. Classical vs. Generalized Witt vectors for nontrivial pro-$p$ $G$

To understand the structure of $\mathbf{W}_G(k)$ one must first understand the frame $\mathcal{F}(G)$ defined earlier. The frame $\mathcal{F}(\mathbf{Z}_p)$ is fairly simple. The finite transitive $\mathbf{Z}_p$-sets are $\mathbf{Z}_p/p^n\mathbf{Z}_p$ for $n \ge 0$. The set $\mathcal{F}(\mathbf{Z}_p)$ is totally ordered since there is a $\mathbf{Z}_p$-map from $\mathbf{Z}/p^n\mathbf{Z}$ to $\mathbf{Z}/p^m\mathbf{Z}$ if and only if $n \ge m$. Since $\mathcal{F}(G)$ is a partially ordered set, one may draw a graphical representation of $\mathcal{F}(G)$ called its Hasse diagram. Figure 1 is part of the Hasse diagram for $\mathcal{F}(\mathbf{Z}_2)$ and Figure 2 is part of the Hasse diagram for $\mathcal{F}(\mathbf{Z}_2^2)$ with a particular pair of elements $U$ and $T$ in $\mathcal{F}(G)$ labeled. Dots represent elements of $\mathcal{F}(G)$, where a line is drawn between two elements $T$ and $U$ provided $U < T$ and there are no elements in between.

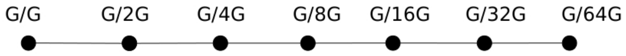

FIGURE 1. Portion of the frame $\mathcal{F}(\mathbf{Z}_2)$

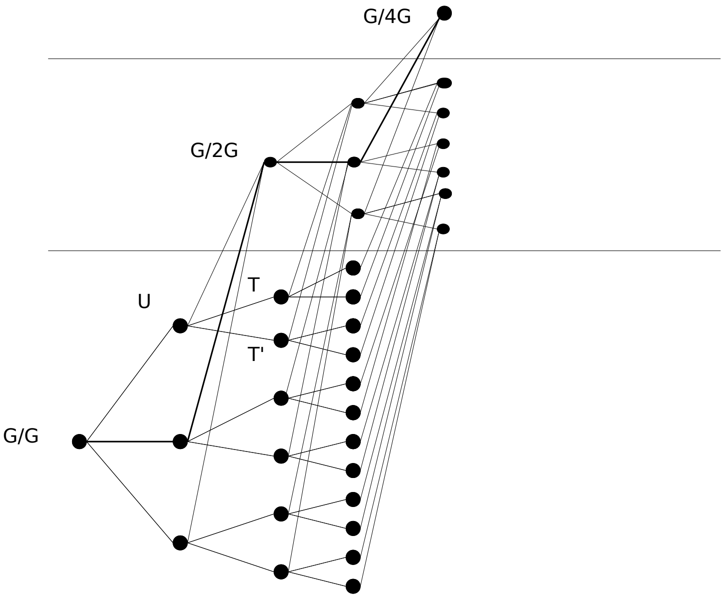

FIGURE 2. Portion of the frame $\mathcal{F}(\mathbf{Z}_2^2)$

The frame of $\mathbf{Z}_2^2$ is far more complicated than the frame of $\mathbf{Z}_2$. Figure 2 appears similar to the tree of $\mathbf{Z}_2$-lattices in $\mathbf{Q}_2^2$ up to scaling [31, p. 70-71] since each $T \in \mathcal{F}(\mathbf{Z}_2^2)$ has the form $\mathbf{Z}_2^2/H$ for a unique $\mathbf{Z}_2$-lattice $H$ in $\mathbf{Z}_2^2$, but there are some fundamental differences between Figure 2 and the tree of $\mathbf{Z}_2$-lattices. For example, $\mathbf{Z}_2^2/\mathbf{Z}_2^2$ and $\mathbf{Z}_2^2/2\mathbf{Z}_2^2$ appear as different nodes in Figure 2 but $\mathbf{Z}_2^2$ and $2\mathbf{Z}_2^2$ are the same lattice up to scaling. Also, $\mathcal{F}(\mathbf{Z}_2^2)$ is not homogeneous (the trivial $\mathbf{Z}_2^2$-set only has 3 edges coming out of it, every other vertex has more).

A particularly important aspect about $\mathcal{F}(\mathbf{Z}_p^d)$ when $d \geq 2$ is there is more than one $G$-set covering the trivial $G$-set. In general, when $G$ is any pro-$p$ group that is not pro-cyclic there is more than one maximal open subgroup $H$, necessarily normal, which leads to multiple transitive $G$-sets $G/H$ in $\mathcal{F}(G)$ with size $p$. To see how this plays a role let's recall how multiplication by $p$ is determined in the classical Witt vectors $\mathbf{W}_{\mathbf{Z}_p}(k)$. For $\mathbf{a} \in \mathbf{W}_{\mathbf{Z}_p}(k)$, with $\mathbf{a} = (a_0, a_1, a_2, \ldots)$, $p\mathbf{a} = (0, a_0^p, a_1^p, \ldots)$. The effect on the coordinates is just shifting and raising to the $p$-th power. If $G$ is not pro-cyclic and $\mathbf{a} \in \mathbf{W}_G(k)$, then the $T$th coordinate of $p\mathbf{a}$ for $\#T = p$ is $a_0^p$ and there is more than one such

$T$. So there are repetitions in the coordinates of $p\mathbf{a}$ and this shows the ideal $(p)$ is smaller than the maximal ideal $\mathfrak{m}$ of $\mathbf{W}_{\mathbf{Z}_p^d}(k)$, unlike the classical case. One might expect that if $G$ is pro-$p$ and topologically finitely generated and $k$ is a perfect field of characteristic $p$ then $\mathbf{W}_G(k)$ is Noetherian and $\mathfrak{m}$ is generated by the finitely many vectors of the form $(0, \ldots, 0, 1, 0, \ldots)$ where the single 1 is in a coordinate indexed by $T \in \mathcal{F}(G)$ of size $p$. This turns out to be false and $k$ being perfect plays no role in the difficulties.

**Theorem 1 (M).** *When $G = \mathbf{Z}_p^d$ with $d \geq 2$ and $k$ is a field of characteristic $p$, the maximal ideal $\mathfrak{m}$ of $\mathbf{W}_G(k)$ is not finitely generated, so $\mathbf{W}_G(k)$ is not Noetherian. In fact $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \infty$.*

How does one prove this? We look at the coordinates of elements of $\mathfrak{m}^2$. For every $n \geq 1$ there are non-isomorphic $G$-sets $T_n$ and $T_n'$ in $\mathcal{F}(G)$ of size $p^n$ such that

$$\{U : U < T_n\} = \{U : U < T_n'\}. \tag{1}$$

This is illustrated in Figure 2 with $T_2 = T$ and $T_2' = T'$. For $T_n$ and $T_n'$ satisfying (1) I showed all elements of $\mathfrak{m}^2$ have the *same* $T_n$ and $T_n'$ coordinates. It follows from this that Witt vectors of the form $\mathbf{a}_n = (0, \ldots, 0, 1, 0, \ldots)$ for $n \geq 1$, where the nonzero coordinate is indexed by $T_n$ (one index from the pair $T_n$ and $T_n'$ in (1)), are all in $\mathfrak{m}$ but not $\mathfrak{m}^2$. I showed the $\mathbf{a}_n$'s are $k$-linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, so $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \infty$ which shows $\mathfrak{m}$ is not finitely generated. This was actually my second proof of Theorem 1. My first proof of the theorem proceeded by looking at equations involving the coordinates of elements in a proper finitely generated ideal, resulting in a contradiction when applied to a general element of $\mathfrak{m}$ if $\mathfrak{m}$ were finitely generated. I also proved Theorem 1 for some nonabelian $G$.

Another consequence of there being more than one maximal subgroup of an infinite pro-$p$ group $G$ when $G \not\cong \mathbf{Z}_p$ is the ability to easily construct zero divisors. In particular,

**Theorem 2 (M).** *For any nontrivial pro-$p$ group $G \not\cong \mathbf{Z}_p$ and field $k$ of characteristic $p$, the ring $\mathbf{W}_G(k)$ is not a domain.*

At this point one may ask, what known "$p$-adic" rings are local of characteristic 0, non-domains and non-Noetherian? The ring of continuous functions $C(\mathbf{Z}_p, \mathbf{Z}_p)$ is of characteristic 0, not a domain and non-Noetherian. Of course it is not local, however one can localize $C(\mathbf{Z}_p, \mathbf{Z}_p)$ at a prime ideal such as $\mathfrak{p} = \{f \in C(\mathbf{Z}_p, \mathbf{Z}_p) \colon f(0) \equiv 0 \bmod p\}$. This localization has residue field $\mathbf{F}_p$ and is reduced. With this in mind I studied whether $\mathbf{W}_G(k)$, which is not a domain, is at least reduced.

**Theorem 3 (M).** *For any field $k$ of characteristic $p$, the ring $\mathbf{W}_{\mathbf{Z}_p^d}(k)$ is reduced for $d \geq 2$.*

Although I pointed out before that $\mathcal{F}(\mathbf{Z}_2^2)$ in Figure 2 is not the tree of $\mathbf{Z}_2$-lattices in $\mathbf{Q}_2^2$ up to scaling, the proof of Theorem 3 uses combinatorial properties of $\mathcal{F}(\mathbf{Z}_p^d)$ that are reminiscent of the tree structure of the $\mathbf{Z}_p$-lattices in $\mathbf{Q}_p^d$ up to scaling. There are limitations to how far Theorem 3 can be extended: I have found some $G$ where $\mathbf{W}_G(k)$ is not reduced, for example $G = \mathbf{Z}_p \times \mathbf{Z}/p\mathbf{Z}$.

**Future Research (Witt-Burnside Rings):**

**Question 1.** What are the prime ideals in $\mathbf{W}_G(k)$ when $G$ is a pro-$p$ group and $k$ is a field of characteristic $p$? For $G = \mathbf{Z}_p^d$, $d \geq 2$, there is a natural collection of prime ideals in $\mathbf{W}_G(k)$

other than the maximal ideal, suggested by the picture of $\mathcal{F}(G)$. However the intersection of these prime ideals is not zero and since $\mathbf{W}_G(k)$ is reduced in this case there must be more prime ideals.

**Question 2.** Is $\mathbf{W}_G(k)$ complete with respect to its $\mathfrak{m}$-adic topology? It is complete in a natural profinite topology when $G = \mathbf{Z}_p^d$, but the $\mathfrak{m}$-adic topology is different from the profinite topology when $d \geq 2$ on account of elements of $\mathfrak{m}^n$ having specific coordinate repetitions for $n \geq 1$. It is difficult to study $\mathfrak{m}$-adic convergence since it is difficult to see when an element is in $\mathfrak{m}^n$ for $n \geq 1$.

**Question 3.** Do the rings $\mathbf{W}_G(k)$, especially $\mathbf{W}_G(\mathbf{F}_p)$, have some concrete description in terms of known non-Noetherian "$p$-adic" rings? (Note $\mathbf{W}_{\mathbf{Z}_p^2}(k)$ is not $\mathbf{W}_{\mathbf{Z}_p}(\mathbf{W}_{\mathbf{Z}_p}(k))$ since the latter ring is a domain and $\mathbf{W}_{\mathbf{Z}_p^2}(k)$ is not $\mathbf{W}_{\mathbf{Z}_p}(k)^2$ since the latter ring is Noetherian)

**Question 4.** Is there a universal mapping property describing $\mathbf{W}_G(k)$ for perfect fields $k$ of characteristic $p$? The classical Witt vectors over a perfect field $k$ are described via a universal mapping property among $p$-adically complete discrete valuation rings with residue field $k$.

**Question 5.** Is there a characterization of pro-$p$ $G$ such that $\mathbf{W}_G(k)$ is reduced? Some natural questions from commutative algebra are: When is $\mathbf{W}_G(k)$ coherent? Considering the description of the Cohen-Macaulay property for non-Noetherian rings [11], when is $\mathbf{W}_G(k)$ Cohen-Macaulay?

## Manifold Approximation

My computational geometry thesis focuses on constructing piecewise linear approximations to curves and surfaces. On a curve or surface, any finite collection of points can be made into a piecewise linear approximation where we think of the finite set of points as sample points. These are useful in areas such as computational chemistry for modeling molecules and for problems where one needs to solve a PDE over a surface where it is common to use the finite element method, and the efficiency of the triangulation of the surface is an important issue.

From a set of samples, piecewise linear approximations of the manifold can be built which satisfy the following principles: the approximation must be *i*) "close" to the manifold being approximated, *ii*) efficient so as to use the fewest number of linear approximating pieces to approximate the manifold, and *iii*) "topologically equivalent" to the original manifold.

As one insists on arbitrarily close approximations, the number of approximating pieces used gets arbitrarily large. However, one should expect the number of approximating pieces to be asymptotically bounded by some function of the curvature of the manifold being approximated. Computational topology is an emerging subfield of computational geometry which includes topological considerations [1, 2, 20, 27]. A property of interest in many applications, for example modeling strands of DNA, is whether or not the approximation is "topologically equivalent" to the manifold. The notion of topological equivalence of most use to manifold approximation is ambient isotopy.

**Computational Geometry of Surfaces.** Surface approximations based on a sufficiently dense sample set are well-known [6, 19, 34]. My focus is to give an efficient construction of such sample sets using sets with prescribed discrepancy with respect to $\sqrt{\kappa}$, where $\kappa$ is the Gauss curvature of the surface. In particular, I utilize the theory of low-discrepancy sequences to construct sample sets with strong enough pseudorandom properties to appropriately saturate level sets of the surface according to curvature. The novel extension here is the flexibility of sets of low discrepancy to handle the approximating geometry which more closely reflects the local curvature on the manifold. While sufficiently dense sample sets for surface approximation have been studied, this is the first explicit construction of such sets that has appeared in the literature.

The main result I provide for surfaces builds on work by Liebon and Letscher [19] and Clarkson [6]. These works give precise, quantitative connections between the Hausdorff distance $\eta$ between the manifold and the approximation and the size of the sample set. I show a similar sample set can be constructed efficiently requiring only a polynomial number (in $\int \sqrt{\kappa}$ and $\eta^{-1}$) of evaluations of the surface and its derivatives. The asymptotic dependence on curvature is tight, whereas our asymptotic dependence on the approximation parameter $\eta$ is tight only up to a $\log \eta^{-1}$ term, arising from the pseudorandom construction of low-discrepancy sets.

**Theorem 4 (M et al.).** *Let $S$ be a surface, $\beta$ be the maximum curvature of $S$ and $\eta$ be small enough so that $\eta(\ln \eta^{-1})^4 \le 9/(32\beta^5)$ and so that $\eta \le \min\{e^{1/\beta^2}, 9(\mathrm{inj}(S)/5)^2, 9(\pi/5\beta)^2\}$ where* inj *is the injectivity radius of the surface. Let $\Lambda$ be a bound on the magnitude of the directional derivative over $S$. Then there is a triangulation of $S$ with Hausdorff distance no more than $\eta$ using at most*

$$C_1(\Lambda)\frac{(\ln(\eta^{-1}))^2}{\eta}\int_S \sqrt{\kappa} + C_2(\beta,\Lambda)\frac{\left(\ln \eta^{-1}\right)^{3/2}}{\eta} \;=\; C_1(\Lambda)\frac{(\ln(\eta^{-1}))^2}{\eta}(1+o(1))\int_S \sqrt{\kappa}$$

*vertices, where $C_1(\Lambda)$ depends only on $\Lambda$ and $C_2(\beta,\Lambda)$ depends only on $\beta$ and $\Lambda$.*

Theorem 4 gives the number of surface elements used to approximate a surface to within $\eta$ to be at most $O(\eta^{-1}\left(\ln \eta^{-1}\right)^2 \int_S \sqrt{\kappa})$.

Just being close is not always sufficient to obtain a homeomorphic approximation. Using the analysis by Dai, Luo, Yau and Gu [34] we can show that the approximation from Theorem 4 is homeomorphic as well. In particular, controlling the deviation between the normal field of the surface and of the triangulation [34, Theorem 3, p. 12] shows that the surface and the triangulation are homeomorphic. The net effect of the estimate we provide is a change in the leading constant of Theorem 4 to be dependent on $\beta$. I hope to show that this homeomorphism is in fact an ambient isotopy as well.

**Computational Topology of Curves.** We address approximation of curves by first constructing sample sets upon which neighborhoods and approximations of curves that are locally adaptive to curvature can be constructed [16, 23]. Thus far my research has resulted in the publications [16, 17, 23] with K. Jordan, E. Moore, T. Peters and A. Russell. The main step is to take as input a subset of the parametric interval which is uniformly distributed with respect to the curvature of the curve $c$. This is a set of low discrepancy with respect to a measure $\mu_c$ where $\mu_c([a,b]) = \int_a^b ||c''(t)||d\lambda$. In particular, more samples will be selected from areas of high curvature and fewer samples from areas of low curvature. Constructing such sets can be done using van der Corput sequences [18, 21]. The main result concerning approximation is

**Theorem 5 (M et al.).** *Let* $c : [0,1] \to \mathbf{R}^3$ *be a* $C^3$*-curve and denote by* $\mu_c$ *the measure defined by estimating the size of a set by the integral of curvature. There exists* $\varepsilon > 0$ *such that when* $X = \{p_1, \ldots, p_n\} \subset [0,1]$ *is a set with discrepancy* $D_{\mu_c}(X) < \varepsilon$*, the polyline over* $\{c(p_i) : i = 1, \ldots, n\}$ *is an ambient isotopic piecewise linear approximation to* $c([0,1])$*.*

The value of Theorem 5 is the weakness of the hypothesis. Previous topological results [27] have primarily given explicit constructions of isotopic approximations for spline curves. This is built from an iterative reduction of the control polygon. The approach I provide using low discrepancy sequences can be applied to a wide class of parametric curves.

**Future Research (Manifold Approximations):**

**Question 1.** Can this approximation be extended to manifolds of higher dimension? Such a generalization still has applications to computational PDEs.

**Question 2.** Is the nearest neighbor mapping from a surface to the approximation I provide an ambient isotopy? Showing the surface approximation I provide is homeomorphic used a strong hypothesis involving principal curvature. It is likely the dependence on principal curvature can be removed and following techniques as in [2], I hope to show this homeomorphism is in fact an ambient isotopy.

**Question 3.** Can the techniques used here be adjusted to take more information into account than curvature? There are two parameters which are used to control when a piecewise linear approximation to a curve is ambient isotopic; one is curvature and the other is a parameter called the minimum separation distance [20]. This second parameter is often approximated computationally using Newton's method, which is not guaranteed to converge. While the explicit computation of this parameter is not needed for the approximation I provide, so far there is not a good method of calculating this parameter.

**Question 4.** Can we reduce a given sample set to a curve or surface to be efficient with respect to curvature? Most often in applications, manifolds come sampled. It would be of interest to use the results I provide to describe an efficient pre-processing algorithm for data input that reduces a sample set which gives an ambient isotopic approximation to only a part of the sample set that is necessary for the approximation to remain ambient isotopic.

## Incidence Algebras

In addition to these two dissertations, I have also researched problems involving incidence algebras. These algebras are of interest in combinatorics because functions such as Moebius functions are elements of incidence algebras. An incidence algebra $I(X, R)$ over a given ring $R$ is the collection of functions $f : X \times X \to R$ where $X$ is a locally finite partially ordered set and $f(x, y) = 0$ for $x > y$. When $X$ is finite, $I(X, R)$ is naturally isomorphic to a subalgebra of an upper triangular matrix algebra over $R$. Group gradings have been studied for matrix algebras [3, 7, 35, 36]. In particular, when $G$ is a torsion-free group, the only $G$-gradings on a matrix algebra are good gradings. Motivated by the study of gradings on the ring of upper triangular matrices, with E. Spiegel, I investigated the possible equivalence of different group gradings on incidence algebras [24].

Given a commutative ring $R$, an $R$-algebra $A$ and a group $G$, a $G$-grading of $A$ is a collection of $R$-modules $A_g$ such that $A = \bigoplus_{g \in G} A_g$ and $A_g A_h \subset A_{gh}$. The elements of $A_g$ are called homogeneous of order $g$, where $g \in G$. Two gradings $A = \bigoplus_{g \in G} A_g = \bigoplus_{g \in G} B_g$ are equivalent if there is an $R$-algebra automorphism $\varphi$ of $A$ such that $\varphi(A_g) = B_g$ for each $g \in G$.

The case when $A$ is an $n \times n$ matrix algebra over $R$ has been considered before [3, 7]. In this case, a $G$-grading is called a *good grading* provided each matrix $e_{i,j}$ is homogeneous, where $e_{i,j}$ is the matrix with a 1 in the $(i, j)$-slot and zeros elsewhere. One way to construct good gradings is to take a $n$-tuple $(g_1, \ldots, g_n) \in G^n$ and give $A$ the grading where $e_{i,j} \in A_{g_i^{-1} g_j}$ and extending linearly. A natural question is whether or not all good gradings arise this way. It has been determined in the case when $R$ is a field that these are the only ones [3]. A. Valenti asked the same question of upper triangular matrices and showed that when $R$ is an algebraically closed field of characteristic 0 and $G$ is a finite abelian group, all good $G$-gradings arise in the manner shown. Motivated by this, with E. Spiegel [24] I studied the group gradings that can exist on $I(X, R)$ based on conditions on the ordered set $X$. In this setting a good grading is one in which each function $e_{x,y}$ is homogeneous, where $e_{x,y}(a, b) = 1$ provided $(x, y) = (a, b)$ and $e_{x,y}(a, b) = 0$ otherwise.

**Theorem 6 (M, Spiegel).** *If $X$ is a bounded countable locally finite partially ordered set and $R$ is an integral domain, and $G$ is a group with the property that the order of any nontrivial torsion element of $G$ is not a unit in $R$, then any $G$-grading of $I(X, R)$ is equivalent to a good grading.*

**Future Research (Incidence Algebras):** A question that would be interesting to study is: to what extent can the coefficient ring $R$ be recovered from $I(X, R)$? That is to say, under what conditions does an isomorphism from $I(X, R)$ to $I(X, S)$ give an isomorphism between $R$ and $S$? Results are known to be positive for semiperfect rings [10], and for rings with trivial lower nil radical [33], provided the ordered set involved satisfies a condition similar to having a waist (having an element comparable with all other elements). A natural extension of this problem is to examine the recovery of a coefficient ring under full $n \times n$ matrix rings. This problem, being almost equivalent to weak $n$-cancellation of modules, has proven to be a topic of significant interest, and has been shown to be false in many natural contexts such as prime Noetherian rings. This gives the problem new perspective, as full matrix rings may be too large to distinguish coefficients, whereas $I(X, R)$, having more ideals than the full matrix ring over $R$, may be closer to $R$ and therefore able to describe the ring more accurately.

## References

[1] K. Abe, J. Bisceglio, T. J. Peters, A. C. Russell, D. R. Ferguson, T. Sakkalis. Computational topology for isotopic surface reconstruction, Theoretical Computer Science 365 (3) (2006) 184 – 198.

[2] N. Amenta, T. J. Peters, A. C. Russell. Computational topology: ambient isotopic approximation of 2-manifolds, Theoretical Computer Science 305 (2003) 3–15.

[3] Y. Bahturin, S. Sehgal, M. Zaicev, Group gradings on associated algebras, Journal of Algebra, 241, (2001), 677-698.

[4] Y. Bahturin, M. Zaicev, Involutions on graded matrix algebras, Journal of Algebra, 315, (2007), 527-540.

[5] M. Brun, Witt vectors and equivariant ring spectra applied to cobordism, Proceedings of the London Mathematical Society, 94, 2, (2007), 351-385.

[6] K. Clarkson. Building triangulations using $\varepsilon$-nets STOC 2006: Proceedings of the Thirty-eighth Annual SIGACT Symposium, 2006.

[7] S. Dăscălescu, B. Ion, C. Năstăsescu, J. Montes, Group gradings on full matrix rings, Journal of Algebra, 220, 2, (1999), 709-728.

[8] A. Dress, C. Siebeneicher, The Burnside ring of profinite groups and the Witt vector construction, Advances in Mathematics, 70 (1988), 87-132.

[9] J. Elliott, Constructing Witt-Burnside rings, Advances in Mathematics, 203, (2006), 319-363.

[10] J. Haack, Isomorphisms of incidence rings, Illinois Journal of Mathematics, 28, 4, (1994), 676-683.

[11] T. Hamilton, T. Marley, Non-Noetherian Cohen-Macaulay rings, Journal of Algebra 307 (2007), 343-360.

[12] M. Hazewinkel, Formal Groups and Applications, Academic Press, New York, 1978.

[13] L. Hesselholt, I. Madsen, On the $K$-theory of finite algebras over Witt vectors of perfect fields, Topology, 36, (1997), 29-101.

[14] M. Hochster, Big Cohen-Macaulay modules and algebras and embeddability in rings of Witt vectors, Queen's papers in Pure Applied Mathematics, 42 (1975), 106-195.

[15] N. Jacobson, Basic Algebra II, 2nd ed., W.H. Freeman and Company, San Francisco, 1989.

[16] K. E. Jordan, L. Miller, E. L. F. Moore, T. J. Peters, A. C. Russell. Modeling time and topology for animation and visualization with examples on parametric geometry, Theoretical Computer Science, 405, 2008, 41-49.

[17] K. E. Jordan, L. Miller, T. J. Peters, A. C. Russell. Geometric topology and visualizing 1-Manifolds, Topological Data Analysis and Visualization: Theory, Algorithms and Applications (to appear).

[18] L. Kupiers, H. Niederreiter. Uniform distribution of sequences, Wiley, New York, 1974.

[19] G. Leibon, D. Letscher. Delauany triangulations and Voronoi diagrams for Riemannian manifolds. SCG 2000: Proceedings of the Sixteenth Annual Symposium on Computational geometry, pages 341-349, New York, NY, USA 2000. ACM Press.

[20] T. Maekawa, N. M. Patrikalakis, T. Sakkalis, G. Yu. Analysis and applications of pipe surfaces, Computer Aided Geometric Design 15 (5) (1998) 437–458.

[21] J. Matousek, Geometric discrepancy, Springer-Verlag, New York, 1999.

[22] N. Metropolis, G-C. Rota, Witt vectors and the algebra of necklaces, Advances in Mathematics, 50 (1983), 2, 95-125.

[23] L. Miller, E. L. F. Moore, T. J. Peters, A. C. Russell. Topological neighborhoods for spline curves : practice & theory, Lecture Notes in Computer Science, 5045, 2008, 149-161.

[24] L. Miller, E. Spiegel, Group gradings on incidence algebras, in Communciations in Algebra (to appear).

[25] L. Miller, On the Structure of Witt-Burnside rings, Ph.D. thesis, University of Connecticut (in progress).

[26] L. Miller, Discrepancy and isotopy for manifold approximations, Ph.D. thesis, University of Connecticut (2009).

[27] E. L. F. Moore, Computational topology of spline curves for geometric and molecular approximations, Ph.D. thesis, University of Connecticut (2006).

[28] W. Nizol, $p$-adic motivic cohomology in arithmetic, International Congress of Mathematicians. Vol. II, 459–472, Eur. Math. Soc., Zürich, 2006.

[29] J.-P. Serre, Local Fields, Springer, New York, 1979.

[30] J.-P. Serre, Algebraic Groups and Class Fields, Springer, New York, 1988.

[31] J.-P. Serre, Trees, Springer, New York, 1977.

[32] E. Spiegel, Involutions in incidence algebras, Linear Algebra and its Applications, 405, (2005), 155-162.

[33] E. Spiegel, The lower nilradical of an incidence ring and applications to the recovery question, Communications in Algebra, 32, 2, (2004), 621-628.

[34] S. Yau, X. Gu, J. Dai, W. Luo. Geometric accuracy analysis for discrete surface approximation, Computer Aided Geometric Design, 24, (2007), pg. 323-338.

[35] A. Valenti, The graded identities on upper triangular matrices of size two, Journal of Pure and Applied Algebra, 172, (2002),325-335.

[36] A. Valenti, M. Zaicev, Abelian gradings on upper triangular matrices, Archiv der Mathematik, 80, (2003), 12-17.